Lipschitz-volume rigidity for metric surfaces

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The Lipschitz-volume rigidity problem

Problem

Let $f : X \to Y$ be a 1-Lipschitz and surjective map between metric spaces that have the same volume. Is f an isometry?

Lipschitz map: $d_Y(f(x), f(y)) \leq L d_X(x,y)$

Theorem (Folklore, Burago–Ivanov, Besson–Courtois–Gallot)

If X, Y are closed Riemannian n-manifolds, then yes.

Extensions to singular settings:

- Alexandrov spaces (Storm, Li)
- Limit RCD spaces (Li–Wang)
- Integral current spaces (Basso-Creutz–Soultanis, Del Nin–Perales, Züst)

Metric surfaces

Metric surface X:

- topological 2-dimensional manifold with a metric
- locally finite area (Hausdorff 2-measure)

Theorem (Meier–N. 2023)

*Let X be a closed metric surface and Y be a closed Riemannian surface with the **same area**. Then every 1-Lipschitz map from X onto Y is an isometry.*

Metric surfaces

Let $f\colon X \to Y$ be 1-Lipschitz and surjective and $\mathcal{H}^2(X) = \mathcal{H}^2(Y)$.

$$\begin{aligned}
\mathcal{H}^2(Y) &= \mathcal{H}^2(f(X)) && \text{(surjective)}\\
&\le \mathcal{H}^2(f(A)) + \mathcal{H}^2(f(X\setminus A)) &&\\
&\le \mathcal{H}^2(A) + \mathcal{H}^2(X\setminus A) && \text{(1-Lipschitz)}\\
&= \mathcal{H}^2(X) && (A \text{ measurable})\\
&= \mathcal{H}^2(Y) && \text{(equal area)}
\end{aligned}$$

Therefore, $\mathcal{H}^2(f(A)) = \mathcal{H}^2(A)$ for each measurable set $A \subset X$.
f is *area-preserving*

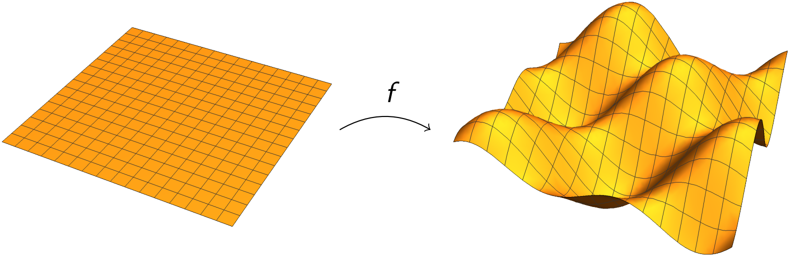
Theorem (Meier–N. 2023)

Let X, Y be metric surfaces without boundary and $f\colon X \to Y$ be area-preserving, 1-Lipschitz, and surjective. If Y is Riemannian, then f is an isometry.

What is special about metric surfaces?

Theorem (Uniformization Theorem, Koebe, Poincaré 1907)

Every simply connected Riemannian surface can be ***conformally*** *uniformized by the complex* ***plane*** *or the unit* ***disk*** *or the Riemann* ***sphere****.*



f **conformal:** balls $\longrightarrow$ balls (or squares $\rightarrow$ squares) in infinitesimal scale

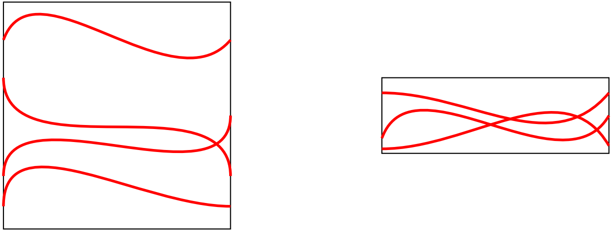
Geometric definition of quasiconformality

X metric surface

Γ family of curves in X

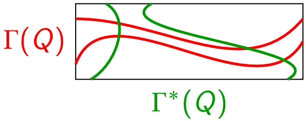
$\rho\colon X \to [0,\infty]$ is *admissible* for Γ if $\int_\gamma \rho\, ds \geq 1$ for all $\gamma \in \Gamma$

$\operatorname{Mod}\Gamma = \inf_\rho \int_X \rho^2\, d\mathcal{H}^2 \longrightarrow$ Outer measure on curve families

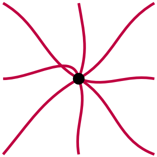


f **conformal:** $\operatorname{Mod}\Gamma = \operatorname{Mod} f(\Gamma)$

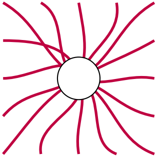
f **quasiconformal:** $K^{-1}\operatorname{Mod}\Gamma \leq \operatorname{Mod} f(\Gamma) \leq K \operatorname{Mod}\Gamma$



$\mathrm{Mod}\,\Gamma(Q)\cdot\mathrm{Mod}\,\Gamma^*(Q)=1$



$\mathrm{Mod}\,\Gamma=0$

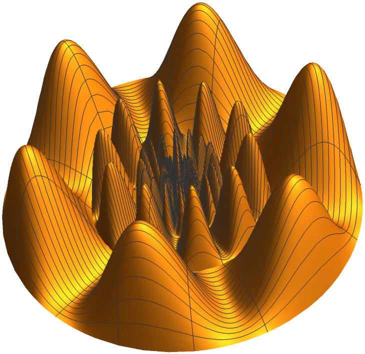


$\mathrm{Mod}\,\Gamma>0$

Quasiconformal uniformization

(Quasi)conformal parametrization $f : \mathbb{C} \to X$

$\Longrightarrow$ The family of (non-constant) curves passing through each point has **modulus zero**

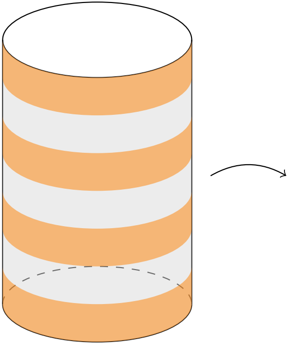


1. Finite area
2. Smooth except for one point P
3. The family of curves passing through P has positive modulus.

$\Downarrow$

No quasiconformal parametrization

Quasiconformal uniformization



Magic Ball
Designed by:
Yuri Shumakov
Presented by:
Jo Nakashima

1. Length-isometric to cylinder outside poles
2. The family of curves through poles has positive modulus
3. Not quasiconformal to sphere

Question

Is this the only enemy?

Quasiconformal uniformization

Question

Is this the only enemy?

Let $C \subset \mathbb{R}^2$ Cantor set. Set $\omega = \chi_{\mathbb{R}^2 \setminus C}$.

$$d_\omega(z,w) = \inf_\gamma \int_\gamma \omega \, ds$$

$(\mathbb{R}^2, d_\omega)$ is homeomorphic to $\mathbb{R}^2$

If $|C| > 0$ then $(\mathbb{R}^2, d_\omega)$ is not quasiconformal to $\mathbb{R}^2$ (Rajala)

Near density points

$$\operatorname{Mod}\Gamma(Q)\operatorname{Mod}\Gamma^*(Q) \to \infty$$

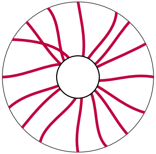
Quasiconformal uniformization

Theorem (Rajala 2017)

Let X be a metric 2-sphere. There exists a ***quasiconformal*** *map $f : \widehat{\mathbb{C}} \to X$ if and only if X is* ***reciprocal****.*

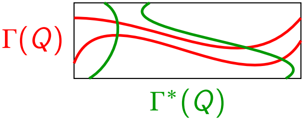
Reciprocity conditions:

(1) The family of non-constant curves passing through each point x has modulus zero.



$$\lim_{r\to 0} \operatorname{Mod} \Gamma(B(x,r), X \setminus B(x,R)) = 0$$

(2) For each topological quadrilateral Q:



$$\kappa^{-1} \le \operatorname{Mod} \Gamma(Q) \cdot \operatorname{Mod} \Gamma^*(Q) \le \kappa$$

Quasiconformal uniformization

- If X is reciprocal, there exists f with $\frac{\pi}{4}\operatorname{Mod}\Gamma \le \operatorname{Mod} f(\Gamma) \le \frac{\pi}{2}\operatorname{Mod}\Gamma$ (Rajala, Romney)
  Optimal constants attained by $id : \mathbb{R}^2 \to X = (\mathbb{R}^2, \ell^\infty)$

- X Ahlfors 2-regular and LLC
  $\Longrightarrow$ Quasiconformal maps are quasisymmetric
  $\Longrightarrow$ Bonk–Kleiner Theorem

- For **every** surface
  $\kappa^{-1} \le \operatorname{Mod}\Gamma(Q)\cdot\operatorname{Mod}\Gamma^*(Q)$ (Rajala–Romney)
  $\kappa^{-1} = (\pi/4)^2$ (Eriksson-Bique–Poggi-Corradini)

- X is reciprocal if and only if
  $\operatorname{Mod}\Gamma(Q)\cdot\operatorname{Mod}\Gamma^*(Q) \le \kappa$ (N.–Romney)

- If the modulus of curves passing through each point is zero, then X is not necessarily reciprocal. (N.–Romney)

Uniformization of arbitrary surfaces

Theorem (N.–Romney 2022)

Every metric surface admits a ***weakly quasiconformal*** *parametrization by a Riemannian surface.*

Corollary

Every metric 2-sphere admits a ***weakly quasiconformal*** *parametrization by the Riemann sphere.*

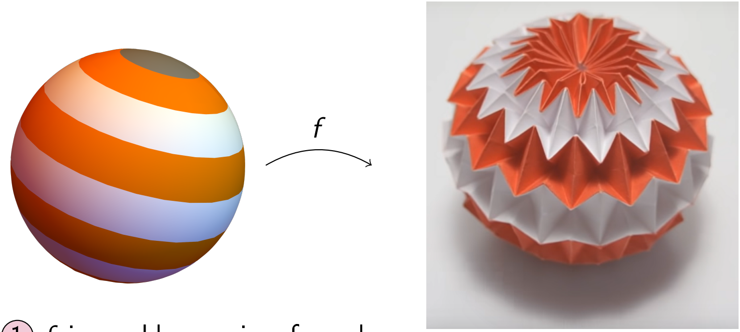
X, Y metric surfaces without boundary
$f: X \to Y$ **weakly quasiconformal**:

- Uniform limit of homeomorphisms
- $\mathrm{Mod}\,\Gamma \leq K \mathrm{Mod}\, f(\Gamma)$

$f: \widehat{\mathbb{C}} \to X$ is QC if and only if X is reciprocal

Earlier versions for locally geodesic and length surfaces (Meier–Wenger 2021, N.–Romney 2021)

Example



1. f is weakly quasiconformal
2. f is **not injective** in black balls around poles
3. f is **conformal** outside black balls

Problem

If the modulus of curves passing through a point $p \in X$ is positive, does there exist a WQC parametrization from a smooth surface that maps a disk to the point p?

Summary

- Riemannian surfaces $\longrightarrow$ conformal parametrization

  (also polyhedral surfaces, Aleksandrov surfaces)

- Reciprocal surfaces $\longrightarrow$ quasiconformal parametrization

  Largest class of surfaces so that modulus in local coordinates is the same as modulus on surface

- Metric surfaces $\longrightarrow$ weakly quasiconformal parametrization

Area-preserving and Lipschitz maps between surfaces

Theorem (Meier–N.)

Let $f: X \to Y$ be area-preserving 1-Lipschitz, and surjective. If Y is Riemannian then f is an isometry.

First step: Preservation of length

Question

If f is area-preserving and Lipschitz, does it quasi-preserve the length of Mod-a.e. path?

$$K^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq K\ell(\gamma)$$

Yes if X is reciprocal (Meier–N.).
Yes if X 2-rectifiable (Creutz–Soultanis).

f need not be injective:
$I = [0,1] \times \{0\}$, $\quad Y = \mathbb{R}^2/I$, $\quad f: \mathbb{R}^2 \to \mathbb{R}^2/I$ projection
Then f is area-preserving and 1-Lipschitz but not injective.

Theorem (Meier–N.)

Let $f: X \to Y$ be area-preserving, Lipschitz, and surjective. If Y is ***reciprocal****, then f is a K-quasiconformal* ***homeomorphism*** *and*

$$K^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq K\ell(\gamma)$$

for Mod-a.e. curve γ. If f is 1-Lipschitz, then $K = 1$.

f does not preserve the length of **every** curve:
Let $\omega(x,0) = x$, $0 \leq x \leq 1$, and $\omega(x,y) = 1$ otherwise. Define

$$d(z,w) = \inf_{\gamma} \int_{\gamma} \omega \, ds$$

The identity $\mathrm{id}: \mathbb{R}^2 \to (\mathbb{R}^2, d)$ is 1-Lipschitz, 1-quasiconformal but

$$\ell_d([0,t]) = t^2/2$$

Area-preserving and Lipschitz maps between surfaces

Y is **upper Ahlfors** 2-**regular** if $\mathcal{H}^2(B(x,r)) \leq Cr^2 \Rightarrow$ Reciprocal

Theorem (Meier–N.)

Let $f: X \to Y$ be area-preserving, Lipschitz, and surjective. If Y is upper Ahlfors 2-regular, then f is a homeomorphism and

$$K^{-1}\ell(\gamma) \leq \ell(f \circ \gamma) \leq K\ell(\gamma)$$

*for **every** curve γ.*

In general we do not have $K = 1$:
Let $\omega(x,0) = 1/2$, $0 \leq x \leq 1$, and $\omega(x,y) = 1$ otherwise. Define

$$d(z,w) = \inf_{\gamma} \int_{\gamma} \omega \, ds$$

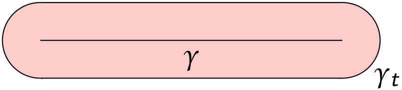
$(\mathbb{R}^2, d)$ is bi-Lipschitz to $\mathbb{R}^2$
but $\ell_d([0,1]) = 1/2 \neq \ell([0,1])$.

Theorem (Meier–N.)

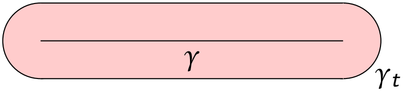
Let $f : X \to Y$ be area-preserving, 1-Lipschitz, and surjective. If Y is Riemannian then f is an isometric homeomorphism.

Proof sketch:

- f is a 1-QC homeomorphism, preserves length of **a.e. curve**.
- Let γ be a curve in Y. Claim: $\ell(f^{-1} \circ \gamma) \leq \ell(\gamma)$.
- γ_t = curve at distance t from γ
- Coarea inequality in Riemannian manifolds: $\int_0^r \ell(\gamma_t)\, dt \leq \mathcal{H}^2(N_r(|\gamma|))$.
- Area bound in Riemannian manifolds: $\mathcal{H}^2(N_r(|\gamma|)) \leq 2r\ell(\gamma) + O(r^2)$ as $r \to 0$.



Proof sketch



We have

$$r \cdot \operatorname*{essinf}_{t\in(0,r)} \ell(\gamma_t) \leq \int_0^r \ell(\gamma_t)\, dt \leq 2r\ell(\gamma) + O(r^2).$$

There exists a sequence $t_n \to 0$ such that

- $\ell(\gamma_{t_n}) \leq 2\ell(\gamma) + o(1)$
- $\ell(\gamma_{t_n}) = \ell(f^{-1} \circ \gamma_{t_n})$

By lower semi-continuity of length

$$2\ell(f^{-1}\circ\gamma) \leq \liminf_{n\to\infty} \ell(f^{-1}\circ\gamma_{t_n}) \leq 2\ell(\gamma).$$

Coarea inequality

Theorem (Federer)

Let $X \subset \mathbb{R}^2$ and $u\colon X \to \mathbb{R}$ be a continuous function in $W^{1,1}_{\mathrm{loc}}(X)$. Then

$$\int \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt = \int_X g \cdot |\nabla u| \, d\mathcal{H}^2$$

for all Borel functions $g\colon X \to [0,\infty]$.

Theorem (Eilenberg)

Let X be a metric space and $u\colon X \to \mathbb{R}$ be a Lipschitz function. Then

$$\int^* \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int_X g \cdot \mathrm{Lip}(u) \, d\mathcal{H}^2$$

for all Borel functions $g\colon X \to [0,\infty]$.

$$\mathrm{Lip}(u)(x) = \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x,y)}.$$

Coarea inequality

How does the coarea inequality generalize to **Sobolev functions in metric spaces**?

ρ is an **upper gradient** of u if

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds$$

for all curves γ and points x, y on γ.

ρ is a (2-)weak upper gradient of u if this is true for Mod-a.e. curve γ.

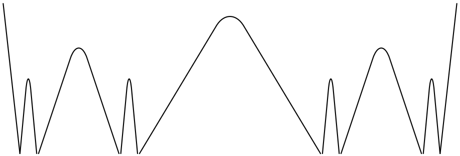
ρ plays the role of $|\nabla u|$

We would like to have a coarea inequality with a **weak upper gradient of** u in place of $|\nabla u|$, $\mathrm{Lip}(u)$

Coarea inequality

Can we have $\int^{*}\int_{u^{-1}(t)} g\, d\mathcal{H}^1\, dt \le \frac{4}{\pi}\int_X g\cdot\rho\, d\mathcal{H}^2$?

No! (Esmayli–Ikonen–Rajala)



- $C\subset\mathbb{R}^2$ Cantor set of positive area
- X metric surface in $\mathbb{R}^3$ containing C such that a.e. curve in X does not "see" C
- $u(x,y,z)=x$ Lipschitz function, weak upper gradient $\rho|_C=0$
- For $g=\chi_C$, $\int_C g\cdot\rho\, d\mathcal{H}^2=0$
- Fubini: $\int\int_{u^{-1}(t)}\chi_C\, d\mathcal{H}^1\, dt=\text{Area}(C)>0$

Theorem (Esmayli–Ikonen–Rajala)

*Let X be a metric surface and $u\colon X \to \mathbb{R}$ be continuous and **monotone** function with a 2-weak upper gradient $\rho \in L^2_{\mathrm{loc}}(X)$. Then*

$$\int^* \int_{u^{-1}(t)} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int_X g \cdot \rho \, d\mathcal{H}^2$$

for all Borel functions $g\colon X \to [0,\infty]$.

Theorem (Meier–N.)

Let X be a metric surface and $u\colon X \to \mathbb{R}$ be continuous function with a 2-weak upper gradient $\rho \in L^2_{\mathrm{loc}}(X)$. Then

$$\int^* \int_{u^{-1}(t)\cap \mathscr{A}_u} g \, d\mathcal{H}^1 \, dt \leq \frac{4}{\pi} \int_X g \cdot \rho \, d\mathcal{H}^2$$

for all Borel functions $g\colon X \to [0,\infty]$.

$\mathscr{A}_u$ = non-degenerate components of level sets of u

Thank you!