Inverse absolute continuity of quasiconformal maps

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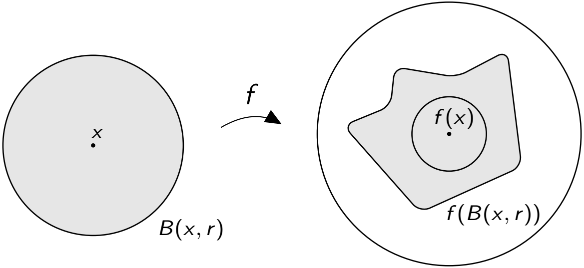
AMS Spring Eastern Sectional Meeting:
Special Session on Analysis, Geometry, and PDEs in Non-smooth Metric Spaces
April 14, 2019

Quasiconformal and Quasisymmetric maps

$U, V \subset \mathbb{R}^n$ open
$f\colon U \to V$ orientation-preserving homeomorphism
f is **quasiconformal** if for each $x \in U$ there exists $r_x > 0$ such that for $r \leq r_x$:



$R_1 \simeq R_2$

f is **quasisymmetric** if the above holds for *all* balls.

Motivation

Question

What is the boundary behavior of quasiconformal maps on the disk/half-space?

$\mathbb{H}$ = Upper half plane
$\partial\mathbb{H} = \mathbb{R}$
Suppose that $f\colon \mathbb{H} \to \mathbb{H}$ is quasiconformal.
f induces a quasisymmetry $f\colon \mathbb{R} \to \mathbb{R}$.

Question

Are $f, f^{-1}\colon \mathbb{R} \to \mathbb{R}$ absolutely continuous in measure?
That is, $|E| = 0 \iff |f(E)| = 0$.

Answer: NO!

Theorem (Ahlfors-Beurling)

There exists a quasiconformal map $f : \mathbb{H} \to \mathbb{H}$ *such that the boundary map* $f : \mathbb{R} \to \mathbb{R}$ *is singular: there exists a set* $E \subset \mathbb{R}$ *with* $|E| = 0$ *such that* $f(E)$ *has full measure.*

Theorem (Tukia)

For each $\alpha > 0$ *there exists a quasisymmetric map* $f : \mathbb{R} \to \mathbb{R}$ *and* $E \subset \mathbb{R}$ *with* $\dim_H(E) < \alpha$ *such that* $\dim_H(\mathbb{R} \setminus f(E)) < \alpha$.

Dimensions $n \geq 2$

Let $f\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+$ be quasiconformal.

f induces a quasisymmetry $f\colon \mathbb{R}^n \to \mathbb{R}^n$.

Quasisymmetries of $\mathbb{R}^n$ are quasiconformal ($n \geq 2$).

Quasiconformal maps $f, f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ are absolutely continuous in measure!

$$|E|_n = 0 \iff |f(E)|_n = 0$$

Main question

Question (Gehring)

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be quasiconformal and let V be an n-dimensional hyperplane (or smooth hypersurface). Are $f|V$, $f^{-1}|f(V)$ absolutely continuous in n-measure?

Answer: If $f(V)$ has locally finite n-measure then $E \subset V$, $|E|_n = 0 \Rightarrow |f(E)|_n = 0$ (Gehring). So f is absolutely continuous in measure.

Partial results

Theorem (Heinonen)

Let $f\colon \mathbb{R}^{n+1}_{+} \to \mathbb{R}^{n+1}$ *be a quasiconformal embedding, extending to* $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_{+}$*. If* $f(\mathbb{R}^n)$ *has a tangent n-plane at a.e. point, then* f *is absolutely continuous in measure:* $E \subset \mathbb{R}^n$, $|E|_n = 0 \Rightarrow |f(E)|_n = 0$.

Theorem (Väisälä)

Let $f\colon \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n$, *be a quasisymmetric embedding such that* $f(\mathbb{R}^n)$ *has locally finite n-measure. Then* $|E|_n = 0 \Rightarrow |f(E)|_n = 0$.

Theorem (David-Semmes, Semmes)

Let $f\colon \mathbb{R}^n \to X$ *be a quasisymmetric homeomorphism and* X *is Ahlfors n-regular* ($|B(x,r)|_n \simeq r^n$) *then* f, f^{-1} *are absolutely continuous in measure:* $|E|_n = 0 \iff |f(E)|_n = 0$.

Problems

Question (Gehring, Heinonen, Astala-Bonk-Heinonen)

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be quasiconformal and let V be an n-dimensional hyperplane (or smooth hypersurface). Is $f^{-1}|f(V)$ absolutely continuous in measure? Can we have $|E|_n > 0$ but $|f(E)|_n = 0$?

Question (Vaïsälä)

If $f : \mathbb{R}^n \to \mathbb{R}^N$, $N \geq n$, is quasisymmetric embedding, is $f^{-1}|f(\mathbb{R}^n)$ absolutely continuous in measure?

Question (Heinonen-Semmes)

If $f : \mathbb{R}^n \to X$ is quasisymmetric homeomorphism, is f^{-1} absolutely continuous in measure? What if we add the assumption that X has locally finite n-measure?

Negative results

Theorem (Romney)

There exists a quasisymmetry $f : \mathbb{R}^n \to X$ *and a set* $E \subset \mathbb{R}^n$ *of full n-measure with* $|f(E)|_n = 0$.

Theorem (Romney)

There exists a quasisymmetry $f : \mathbb{R}^n \to X$, *where* X *has locally finite n-measure, and a set* $E \subset \mathbb{R}^n$ *of positive n-measure with* $|f(E)|_n = 0$.

Back to original problem

Question (Gehring, Heinonen, Astala-Bonk-Heinonen)

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be quasiconformal and let V be an n-dimensional hyperplane (or smooth hypersurface). Is $f^{-1}|f(V)$ absolutely continuous in measure? Can we have $|E|_n > 0$ but $|f(E)|_n = 0$?

Theorem (N.-Romney)

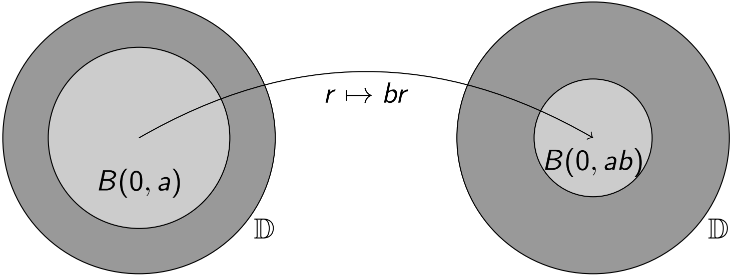
There exists a quasiconformal map $f : \mathbb{R}^3 \to \mathbb{R}^3$ and a set $E \subset \mathbb{R}^2 \times \{0\}$ of full 2-measure such that $|f(E)|_2 = 0$.

Theorem (N.-Romney)

There exists a quasiconformal map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(\mathbb{R}^2 \times \{0\})$ has locally finite 2-measure and there exists a set $E \subset \mathbb{R}^2 \times \{0\}$ with $|E|_2 > 0$ but $|f(E)|_2 = 0$.

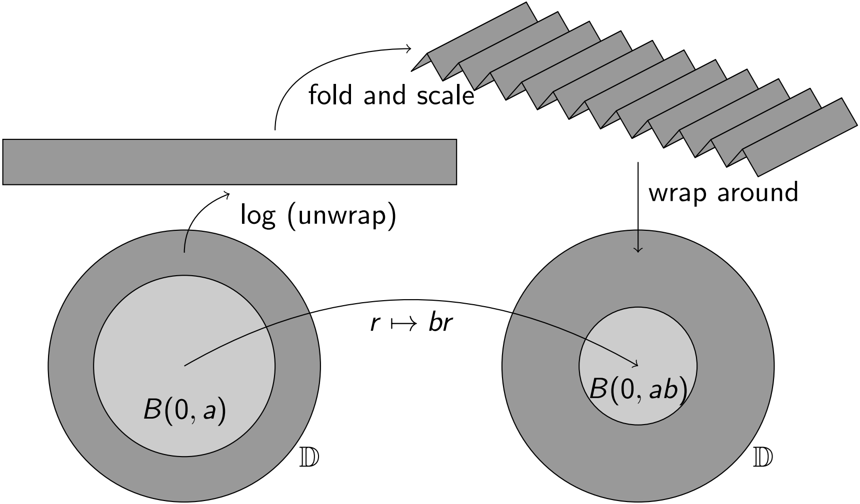
Main ingredient of proof

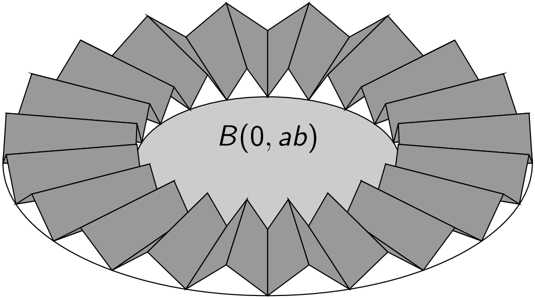
$g\colon \mathbb{D} \to \mathbb{D}$
$B(0,a) \mapsto B(0,ab)$ (conformal)
$A(0;a,1) \mapsto A(0;ab,1)$ (quasiconformal)

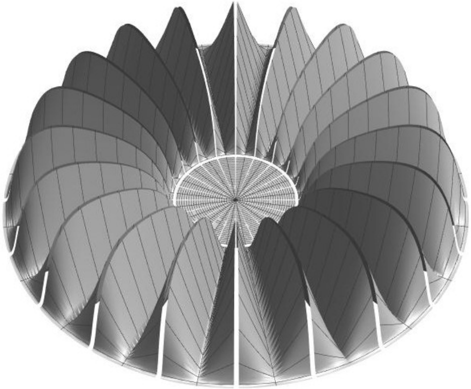


Then g is K-quasiconformal in $\mathbb{D}$ with $K \geq C(a,b) > 1$.

If the target is $\mathbb{R}^3$, then we can have $g\colon \mathbb{D} \to \mathbb{R}^3$ such that:
$B(0,a) \mapsto B(0,ab)$ (conformal)
g is $(1+\varepsilon)$-quasiconformal




$B(0, ab)$



Open Questions

Question

Can such a construction be performed in $\mathbb{R}^n$, $n \geq 3$?

Question

Can f be K-quasiconformal with $K \to 1$? Could it be the case that $(1+\varepsilon)$-quasiconformal maps, restricted to a hyperplane, have absolutely continuous inverse?

Thank you!