(Non)-removability of the Sierpiński gasket

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General problem

$\Omega \subset \mathbb{R}^n$ open set

$$f\colon \overline{\Omega} \to \mathbb{R} \quad (\text{or } \mathbb{R}^n)$$
$$g\colon \mathbb{R}^n \setminus \Omega \to \mathbb{R} \quad (\text{or } \mathbb{R}^n)$$

with $f = g$ on $\partial\Omega$. Define

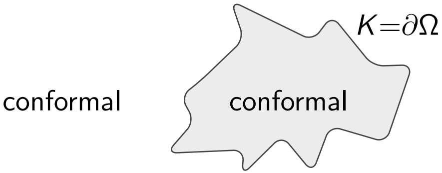
$$h = \begin{cases} f, & \overline{\Omega} \\ g, & \mathbb{R}^n \setminus \Omega \end{cases}.$$

Question: Is h of the same class as f and g?

(Quasi)conformal removability

Definition

Let $K \subset \mathbb{C}$ be a compact set. K is **removable** for (quasi)conformal maps (**QC-removable**) if every homeomorphism $f : \mathbb{C} \to \mathbb{C}$ that is (quasi)conformal in $\mathbb{C} \setminus K$ is (quasi)conformal in $\mathbb{C}$.



Fact: K conformally removable $\iff$ quasiconformally removable

Problem

Find geometric conditions that characterize removability.

Applications:

- Complex Dynamics (quasiconformal surgery) (Shishukura, Sullivan,...)
- Conformal Welding
- Connections to problems of density and extendability of Sobolev functions (Koskela, Rajala, Zhang,...)
- SLE and connection to GFF (Duplantier, Miller, Sheffield,...)

Examples of removable sets

- Sets of σ-finite length (e.g. smooth curves)
- Quasicircles
- Boundaries of John/Hölder domains (quasihyperbolic condition by Jones-Smirnov)

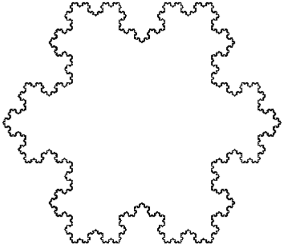
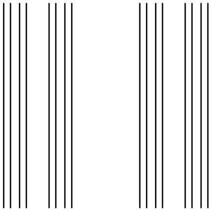
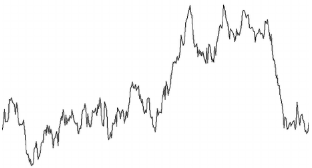


Figure: von Koch snowflake

Examples of non-removable sets

- Sets of positive area
- $C \times [0, 1]$ and some product sets $C \times E$, where C, E are Cantor sets
- Bishop's flexible curves, with Hausdorff dimension 1
- Kaufman's graphs, can be α-Hölder continuous with $\alpha < 1/2$ (Tecu)





Question: Is the graph of Brownian Motion non-removable?

What about sets of more complicated topology?

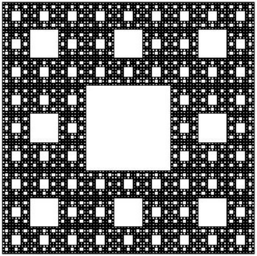


Figure: Sierpiński carpet

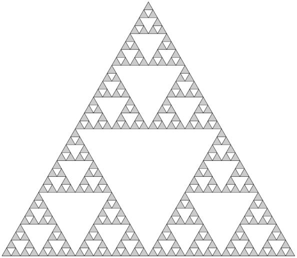


Figure: Sierpiński gasket

Fact: The carpet is non-removable: contains $C \times [0,1]$.
Question:

Is the gasket removable?

Sobolev removability

Definition

Let $p \in [1, \infty]$ and $K \subset \mathbb{C}$ be a compact set. K is $W^{1,p}$-**removable** if any continuous function $f : \mathbb{C} \to \mathbb{R}$ with $f \in W^{1,p}(\mathbb{C} \setminus K)$ lies in $W^{1,p}(\mathbb{C})$.

Facts:

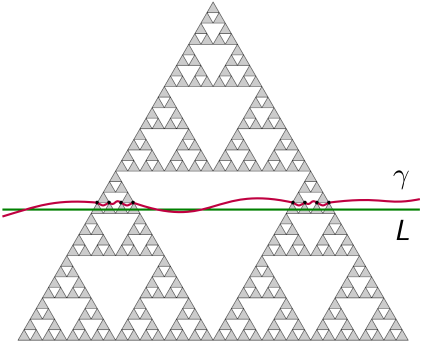
- The problem is local (open question for QC-removability).
- $W^{1,2}$-removable $\implies$ QC-removable (converse?).
- The carpet is non-removable for $W^{1,p}$, $1 \leq p \leq \infty$.

$W^{1,p}$-removability of the gasket

Theorem (N. 2017)

The gasket is $W^{1,p}$-removable for $p > 2$.

Detour property: For each line L there exists a **detour path** γ arbitrarily close to L such that γ intersects only **finitely many** complementary triangles.



Other examples with the detour property

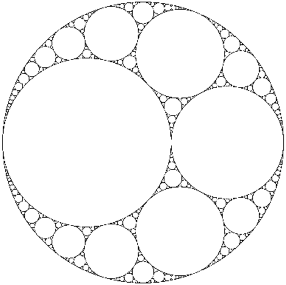


Figure: Apollonian gasket

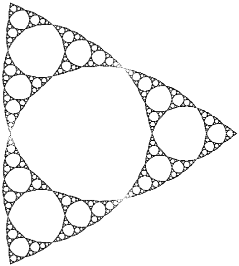


Figure: Julia set of $z^2 - \frac{16}{27z}$

$p = 2$?

Theorem (N. 2018)

All *homeomorhic copies of the gasket are non-removable for* $W^{1,p}$, $1 \leq p \leq 2$.

Corollary

The gasket is $W^{1,p}$*-removable if and only if* $p > 2$.

Question: Is there a topological proof of the non-removability?

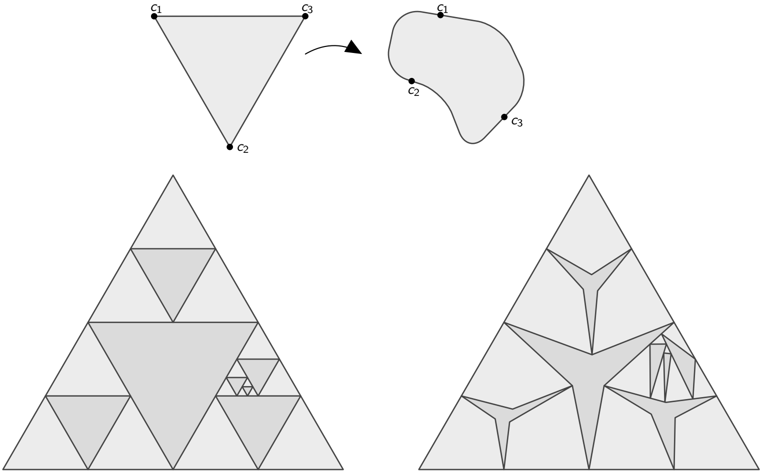
Back to quasiconformal removability

Theorem (N. 2018)

The gasket is non-removable for quasiconformal maps.

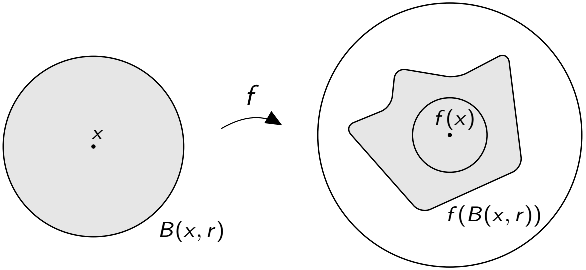
In other words, there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ that is quasiconformal on $\mathbb{C} \setminus K$ but not quasiconformal on $\mathbb{C}$.
Question: What about homeomorphic copies?

Idea



Quasiconformal maps

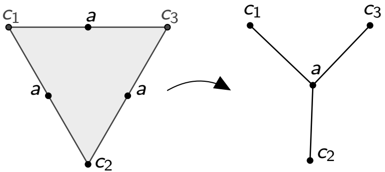
$U, V \subset \mathbb{C}$ open
$f\colon U \to V$ orientation-preserving homeomorphism
f is **quasiconformal** if for each $x \in U$ there exists $r_x > 0$ such that for $r \leq r_x$:



$R_1 \simeq R_2$

f is **quasisymmetric** if r_x does not depend on x.

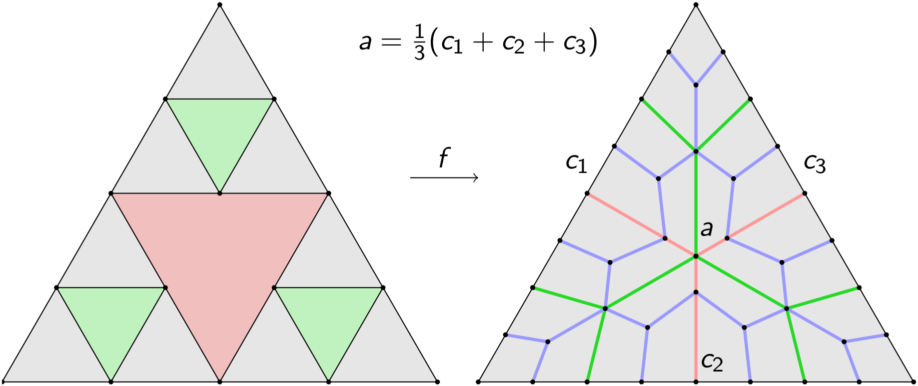
Step 1: Collapse triangles to tripods continuously



$$\begin{aligned} \text{vertices} &\mapsto \text{vertices} \\ \text{midpoints} &\mapsto \text{barycenter} \end{aligned}$$

Step 1: Collapse triangles to tripods continuously

$$a = \frac{1}{3}(c_1 + c_2 + c_3)$$



Obtain a map $f : \mathbb{C} \to \mathbb{C}$:

- continuous and surjective
- injective outside triangles
- $f(K)$ has full Lebesgue measure

Step 2: Create an abstract surface

Create an abstract surface S and "extend" $f\colon \mathbb{C} \to \mathbb{C}$ to a homeomorphism $\Phi\colon \mathbb{C} \to S$ that is quasiconformal outside K.

- $\Phi(K)$ has positive Hausdorff 2-measure in S
- S is 2-**regular**: $\mathcal{H}^2(B(x,r)) \simeq r^2$
- S is a **quasiplane**: there exists a quasisymmetry $\Psi\colon S \to \mathbb{C}$

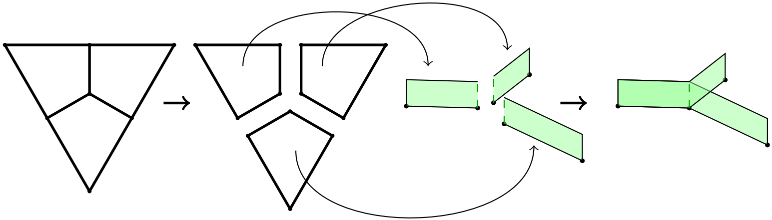
Step 3: The exceptional homeomorphism

$$\mathbb{C} \xrightarrow[\substack{\text{quasiconformal} \\ \text{on } \mathbb{C}\setminus K}]{\Phi} S \xrightarrow[\substack{\text{quasisymmetric} \\ \text{everywhere}}]{\Psi} \mathbb{C}$$

- homeomorphism, quasiconformal on $\mathbb{C} \setminus K$
- not quasiconformal on $\mathbb{C}$:

$$K \mapsto \Phi(K) \quad \text{(positive 2-measure)}$$
$$\Phi(K) \mapsto \Psi \circ \Phi(K) \quad \text{(positive area)}$$

Construction of the surface S: Folding a triangle over a tripod



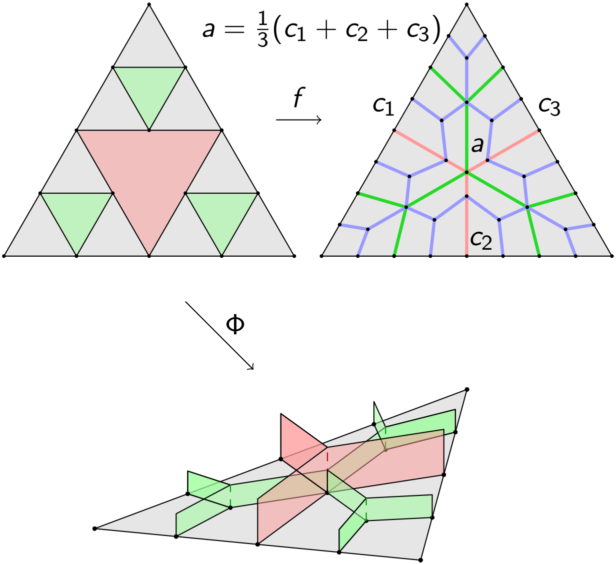
- piecewise linear map
- "extension" of collapsing map f:

$$\begin{aligned} \text{vertices} &\mapsto \text{vertices} \\ \text{midpoints} &\mapsto \text{barycenter} \end{aligned}$$

- M-quasiconformal for a universal M, **independent** of height of rectangles and length of tripod edges

Construction of the surface S: The map Φ

$a = \frac{1}{3}(c_1 + c_2 + c_3)$

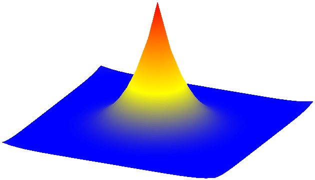


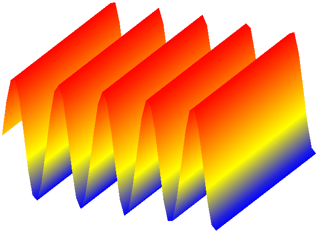
Embedding S to the plane: The map Ψ

Theorem (Bonk-Kleiner 2002)

Let (X, d) *be an* **Ahlfors** 2-**regular** *metric space homeomorphic to the sphere* S^2*. Then* (X, d) *is quasisymmetric to* S^2 *if and only if* (X, d) *is* **LLC***.*

- **Ahlfors 2-regular:** $\mathcal{H}^2(B(x,r)) \simeq r^2$
- **LLC(linearly locally connected):** no *cusps* and no *tall waves/wrinkles*





Thank you!