Optimal bounds for the Beurling-Ahlfors extension operator

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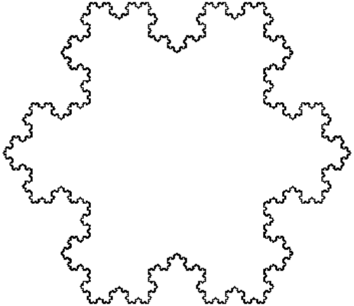
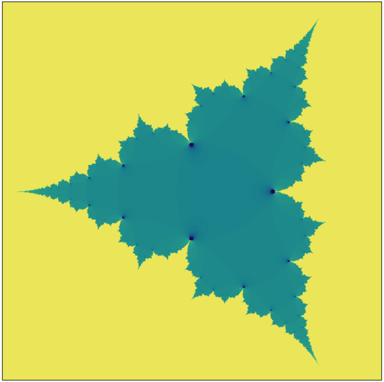
Advanced Courses in Operator Theory and Complex Analysis

June 26–30, 2023

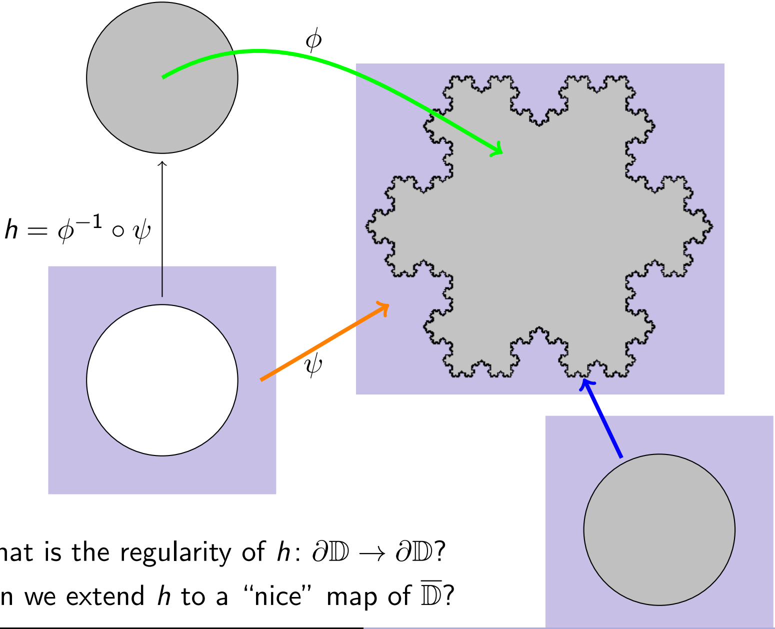
Fractals





Study geometry of fractal curves
by mapping them to unit circle with a "nice" map of $\mathbb{C}$

Constructing "nice" maps

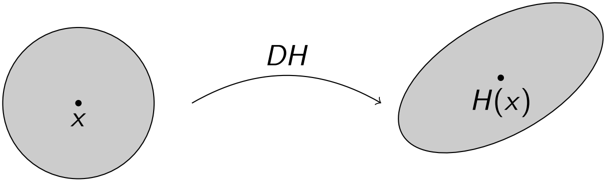


- What is the regularity of $h\colon \partial\mathbb{D} \to \partial\mathbb{D}$?
- Can we extend h to a "nice" map of $\overline{\mathbb{D}}$?

Definitions

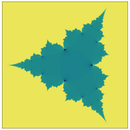
$H\colon U \to V$ homeomorphism between open sets $U, V \subset \mathbb{R}^2$, $H \in W^{1,1}_{\text{loc}}(U)$

$$K_H(x,y) := \inf\{K \geq 1 : \|DH(x,y)\|^2 \leq K J_H(x,y)\} \quad \text{a.e. } (x,y) \in U$$



- H is conformal if $K_H \equiv 1$.
- H is *quasiconformal* if $K_H \in L^\infty(U)$.
- H is a mapping of *finite distortion* if $K_H < \infty$ a.e.
- H is a mapping of *exponentially integrable distortion* if

$$\int_U e^{pK_H(x,y)}\, d\sigma(x,y) < \infty \quad \text{for some } p > 0.$$



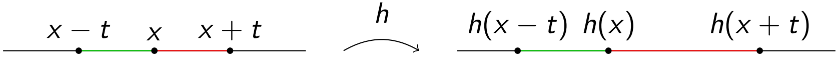
The extension problem

Problem

Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. When does h have an extension to a homeomorphism of $\mathbb{H}$ in one of the previous classes?

For $x \in \mathbb{R}$ and $t > 0$ we define the *symmetric distortion function*

$$\rho_h(x,t) = \max\left\{ \frac{|h(x+t)-h(x)|}{|h(x)-h(x-t)|}, \frac{|h(x)-h(x-t)|}{|h(x+t)-h(x)|} \right\}.$$



Beurling-Ahlfors extension

Theorem (Beurling–Ahlfors, Acta Math. 1956)

There exists an operator that extends h to a C^1-diffeomorphism of $\mathbb{H}$. If $\rho_h(x,t) \leq \varrho$ for some $\varrho > 0$, then $K_h \leq \varrho^2$.

Improved estimates:

- $K_h \leq 8\varrho$ (Reed 1966)
- $K_h \leq 4.2\varrho$ (Li 1983)
- $K_h \leq 2\varrho$ (Lehtinen 1983)

Question

Can we obtain a bound of the form $K_h(x,y) \leq C\rho_h(x,y)$ without any assumptions on ρ_h?

No! (Z. Chen 2001)

Estimates for the dilatation

- (Z. Chen 2001) Under no further assumptions on h,

$$K_h(x,y) \le C\rho_h(x,y)(\rho_h(x+y/2,y/2)+\rho_h(x-y/2,y/2)).$$

Looks like $K_h \le C\rho_h^2$.

- (J. Chen–Z. Chen–He 1996) If $\rho_h(x,t) \le \varrho(t)$ for some decreasing function $\varrho(t)$, then

$$K_h(x,y) \le C\varrho(y/2).$$

- (Zakeri 2008) If $h(x+1)=h(x)+1$ for $x\in\mathbb{R}$ and $\exp(\sup_{y>0}\rho_h(\cdot,y)) \in L^q([0,1])$ for some $q>0$, then

$$K_h(x,y) \le 4\max\left\{\sup_{y>0}\rho_h(x,y), C_1(q)\log\left(\frac{C_2(h)}{y}\right)\right\}.$$

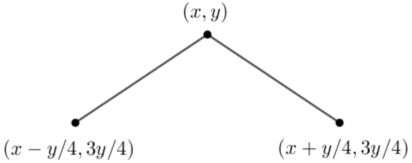
Main result

Theorem (Karafyllia–N., Proc. Lond. Math. Soc. 2022)

Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. Then

$$\frac{\rho_h(x,y)}{4} \le K_h(x,y) \le 50 \max\left\{ \rho_h(x,y), \frac{2}{y}\int_{-y/4}^{y/4} \rho_h(x+z, y-|z|)\, dz \right\}$$

for all $x \in \mathbb{R}$ and $y > 0$.



Consequences:

- $\|\rho_h\|_{L^p} \simeq \|K_h\|_{L^p}$
- $\exp(\rho_h) \in L^p$ for some p if and only if $\exp(K_h) \in L^q$ for some q

Corollaries

Corollary

Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism such that

$$\int_{\mathbb{H}} e^{q\rho_h(x,y)}\, d\sigma(x,y) < \infty$$

for some $q > 0$. Then there exists an extension of h to a homeomorphism of $\mathbb{H}$ that has exponentially integrable distortion.

Corollary

Let $h\colon \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism such that

$$\int_{\mathbb{H}} \rho_h^q(x,y)\, d\sigma(x,y) < \infty$$

for some $q \geq 1$. Then there exists an extension of h to a homeomorphism of $\mathbb{H}$ that has q-integrable distortion.

Necessary conditions

Theorem

Let H be a homeomorphism of $\mathbb{H}$. If $K_H \in L^\infty(\mathbb{H})$ then $\rho_h \in L^\infty(\mathbb{H})$.

Question

Is the converse true? Let H be a homeomorphism of $\mathbb{H}$ that has exponentially integrable distortion. If h is the boundary map, is it true that

$$\int_{\mathbb{H}} e^{q\rho_h(x,y)}\, d\sigma(x,y) < \infty$$

for some $q > 0$?

No! (Koski–Onninen 2022)
Counterexamples for exponentially integrable and q-integrable distortion

Sketch of proof

Let $h\colon \mathbb{R} \to \mathbb{R}$ homeomorphism.

Beurling–Ahlfors extension: $H = (u, v)$

$$u(x,y) = \frac{1}{2y} \int_{x-y}^{x+y} h(t)\, dt$$
$$v(x,y) = \frac{1}{2y} \left(\int_{x}^{x+y} h(t)\, dt - \int_{x-y}^{x} h(t)\, dt \right)$$

We need to find bounds for

$$K_h(x,y) = \frac{\|DH(x,y)\|^2}{J_H(x,y)}.$$

Sketch of proof

Let $x \in \mathbb{R}$ and $y > 0$. Normalize $h^*(0) = 0$, $h^*(1) = 1$. Then

$$K_h(x, y) + \frac{1}{K_h(x, y)} =$$

$$\boxed{F(\xi, \eta) = \frac{1}{\xi + \eta}\left(\beta(1 + \eta^2) + \frac{1}{\beta}(1 + \xi^2)\right)}$$

where $\beta = -h^*(-1), \quad \xi = 1 - \int_0^1 h^*(t)\,dt, \quad \eta = 1 + \frac{1}{\beta}\int_{-1}^0 h^*(t)\,dt.$

Observations (Beurling–Ahlfors):

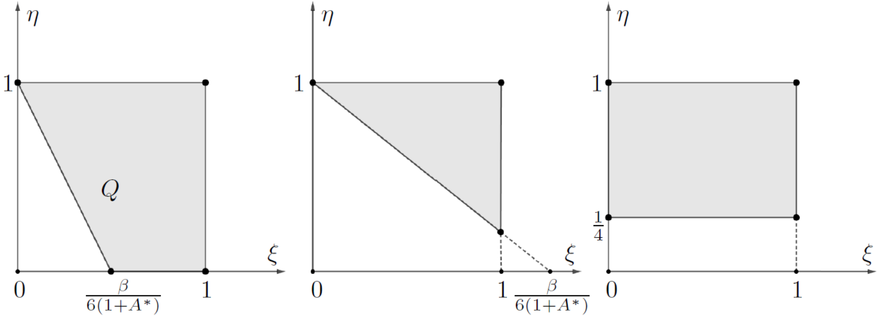
- $\xi, \eta \in (0, 1)$
- F is a **convex** function of ξ, η
- $F(\xi, \eta) \to \infty$ as $\xi, \eta \to 0$.

Sketch of proof

Suppose $\beta \geq 1$ ($\beta < 1$ is symmetric).

There are two cases: $h^*(-1/2) \leq -\beta/2$ and $h^*(-1/2) > -\beta/2$.

(ξ, η) lies in one of the convex polygons:



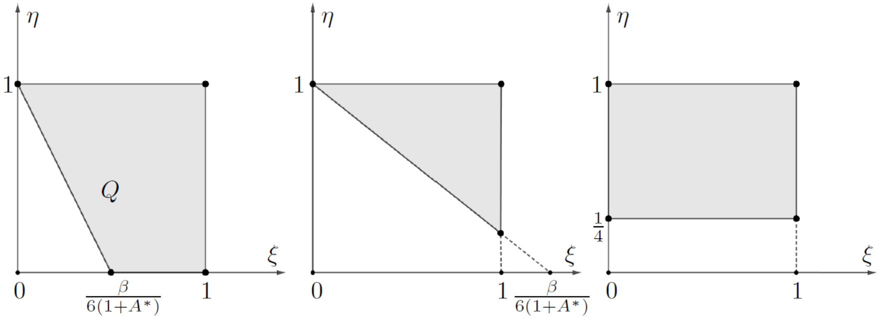
$$A^* = \frac{4}{y} \int_0^{y/4} \rho_h(x + z, y - z) dz.$$

Worst case

$\dfrac{\beta}{6(1+A^*)} < 1 \implies (\xi, \eta)$ lies in the first convex quadrilateral Q.

F is **convex** $\implies$ the maximum in Q is attained at one of the **vertices**

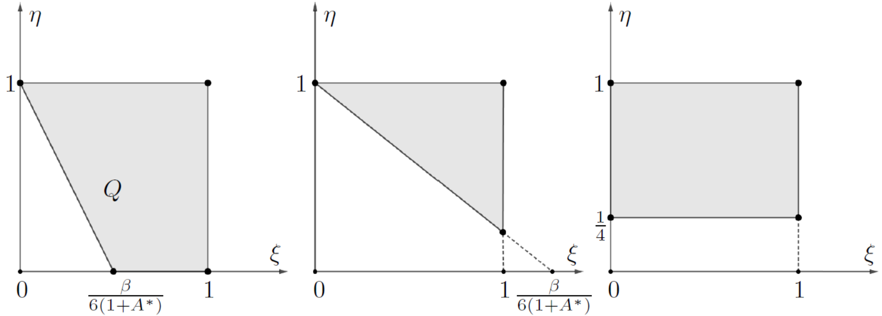
Then $F(\xi,\eta) \leq 25 \max\{\rho_h(x,y), A^*\}$



Better case

$\frac{\beta}{6(1+A^*)} \geq 1 \implies (\xi, \eta)$ lies in the **triangle** with vertices $(0,1)$, $(1,1)$, and $(1,0)$

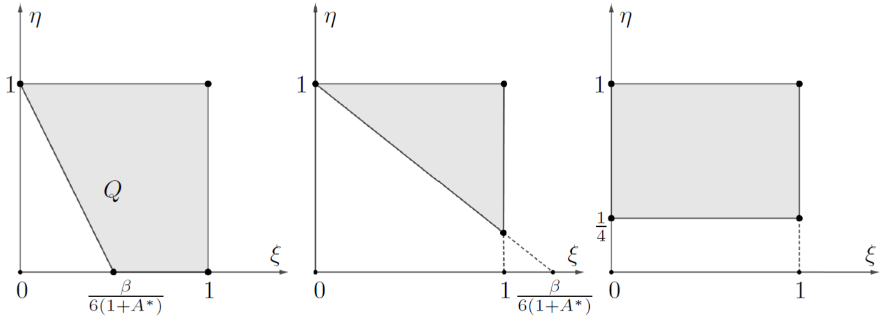
Then $F(\xi,\eta) \leq 3\rho_h(x,y)$.



Sketch of proof

If (ξ, η) lies in the third **rectangle** $0 < \xi < 1$ and $1/4 \leq \eta < 1$, then

$$F(\xi, \eta) \leq 9\rho_h(x, y).$$



Sketch of proof

Combining all cases ($\beta \geq 1$)

$$K_h(x,y) \leq 25 \max \left\{ \rho_h(x,y), \frac{4}{y} \int_0^{y/4} \rho_h(x+z, y-z)\, dz \right\}.$$

The case $\beta < 1$ is symmetric. Thus,

$$K_h(x,y) \leq 50 \max \left\{ \rho_h(x,y), \frac{2}{y} \int_{-y/4}^{y/4} \rho_h(x+z, y-|z|)\, dz \right\}.$$

Thank you for your attention!