Rigidity theorems for circle domains

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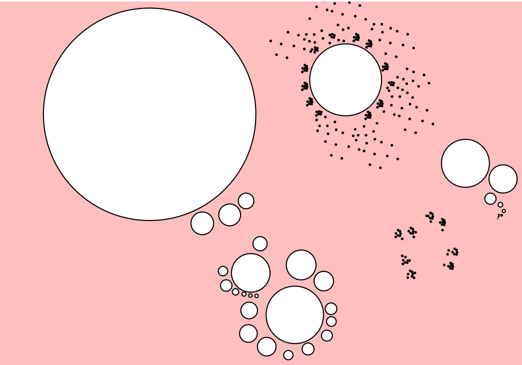
Circle domains

Definition

A domain $\Omega \subset \widehat{\mathbb{C}}$ is a **circle domain** if $\partial\Omega$ consists of points and circles.



The boundary of a circle domain contains at most **countably many circles**.

Koebe's Conjecture

Conjecture (Kreisnormierungsproblem, Koebe 1908)

Every domain Ω is conformally equivalent to a circle domain.

- Simply connected domains: Riemann 1851
- Finitely connected domains: Koebe 1920
- Countably connected domains: He-Schramm 1993
- Uncountably connected domains: Open

Uniqueness

- Finitely connected domains: Koebe 1920
- Countably connected domains: He-Schramm 1993
- Uncountably connected domains: Fails in general!

Observation

If E is a totally disconnected compact set with $\text{Area}(E) > 0$, then there exists a non-Möbius homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that is conformal on $\widehat{\mathbb{C}} \setminus E$.

$$\left.\begin{array}{l}\text{Set } \ \mu = \frac{1}{2}\chi_E \\ \text{Solve Beltrami equation: } \ f_{\overline{z}} = \mu f_z\end{array}\right\} \Rightarrow f$$

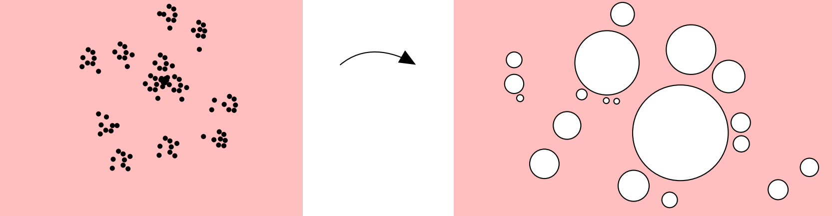
Rigidity

Definition

A circle domain Ω is **conformally rigid** if every conformal map from Ω onto another circle domain is the restriction of a Möbius transformation.

Rigid: Finitely connected, Countably connected

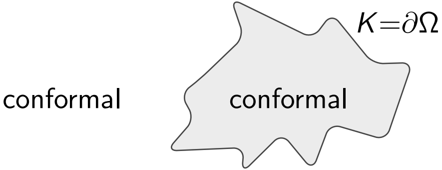
Non-rigid: $\partial\Omega$ is totally disconnected and $\text{Area}(\partial\Omega) > 0$



Removability

Definition

Let $K \subset \widehat{\mathbb{C}}$ be a compact set. K is **conformally removable** if every homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that is conformal in $\widehat{\mathbb{C}} \setminus K$ is conformal in $\widehat{\mathbb{C}}$.



Examples of removable sets

- Sets of σ-finite length (e.g. smooth curves) (Besicovitch 1931)
- Quasicircles
- Boundaries of John/Hölder domains (quasihyperbolic condition by Jones-Smirnov 2000)
- NED sets (Negligible for Extremal Distance), e.g., C, $C \times C$ (Ahlfors-Beurling 1950)

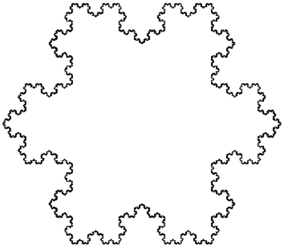
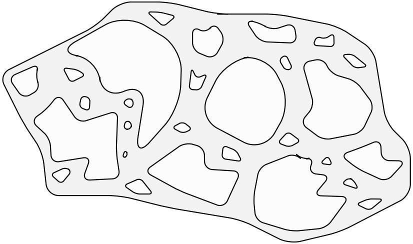
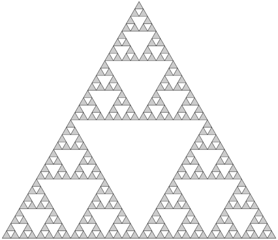


Figure: von Koch snowflake

Examples of non-removable sets

- Sets of positive area
- $C \times [0, 1]$ and some product sets $C \times E$, where C, E are Cantor sets
- Bishop's flexible curves, with Hausdorff dimension 1 (1994)
- Sierpiński carpets (N. 2019)
- Sierpiński gasket (N. 2019)





The Rigidity Conjecture

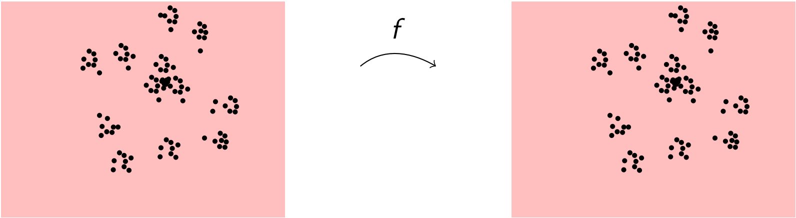
Conjecture (He-Schramm 1994)

Let Ω be a circle domain. The following are equivalent:

(i) *Ω is conformally rigid*

(ii) *$\partial\Omega$ is conformally removable*

If $\partial\Omega$ is totally disconnected, then (i) $\Rightarrow$ (ii).



Known results

	$\partial\Omega$ **removable?**	Ω **rigid?**
Area > 0	N	N (Sibner 1968)
NED	Y (Ahlfors-Beurling 1950)	Y (Ahlfors-Beurling 1950)
finite	Y	Y (Koebe 1920)
countable	Y (Besicovitch 1931)	Y (He-Schramm 1993)
σ**-finite**	Y (Besicovitch 1931)	Y (He-Schramm 1994)
John/Hölder	Y (Jones-Smirnov 2000)	Y (N.-Younsi 2019)
quasihyperbolic	Y (Jones-Smirnov 2000)	Y (N.-Younsi 2019)

The quasihyperbolic condition

$D \subsetneq \mathbb{C}$, $\delta_D(x) = \text{dist}(x, \partial D)$

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{1}{\delta_D} ds$$

Facts:

- $k_D \simeq h_D$ if D is simply connected
- $k_D(x_1, x_2) \simeq \#\{\text{Whitney cubes needed to connect } x_1 \text{ to } x_2\}$

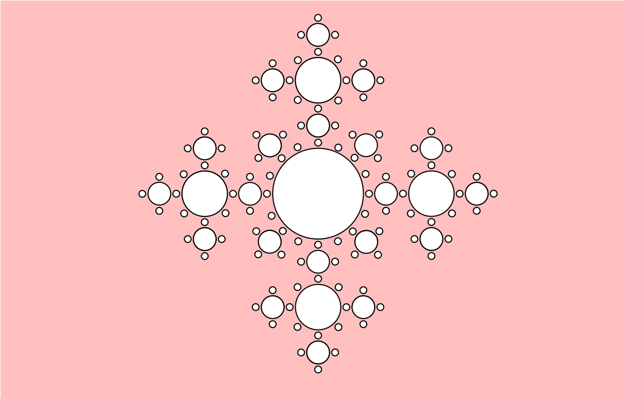
Definition

A domain D satisfies the **quasihyperbolic condition** if $\int_D k_D(x, x_0)^2 dx < \infty$ for some $x_0 \in D$.

Examples: Quasidisks, John domains, Hölder domains

Theorem (N.-Younsi, Invent. Math. 2019)

Let Ω be a circle domain with $\infty \in \Omega$ and consider a ball $B(0,R) \supset \partial\Omega$. If $D = B(0,R) \cap \Omega$ satisfies the quasihyperbolic condition, then Ω is rigid.



Proof

$\Omega, \Omega^* \subset \widehat{\mathbb{C}}$ circle domains, containing ∞
$f : \Omega \to \Omega^*$ conformal

Step 1: f extends to a homeomorphism $\overline{\Omega} \to \overline{\Omega^*}$

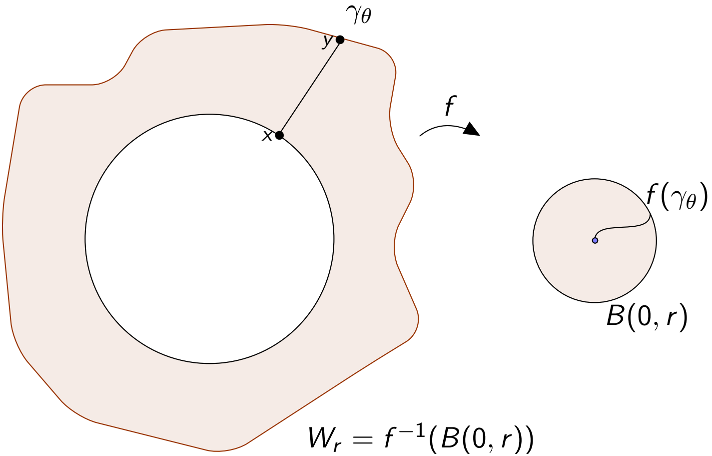
Step 2: f extends to a homeomorphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$

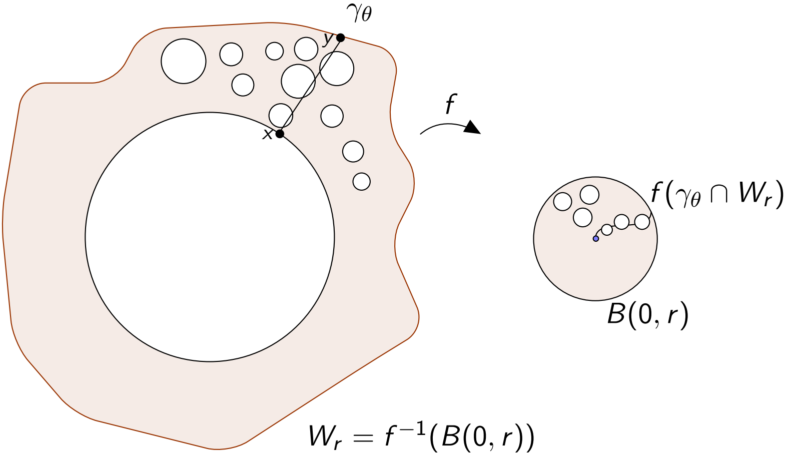
Step 3: f is K-quasiconformal on $\widehat{\mathbb{C}}$, $K = K(\Omega)$

Step 4: $K = 1$ so f is Möbius

Step 1: Extension to $\partial\Omega$

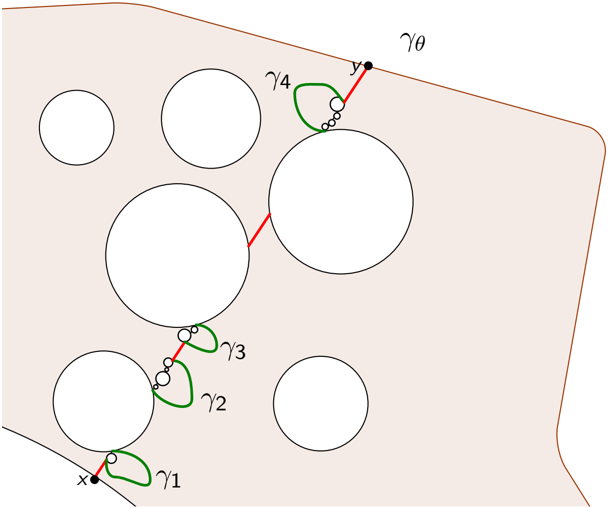
- f does not map boundary circles (continua) to points
- f does not map boundary points to circles (continua)





Finitely many circles:

$$r \leq \int_{\gamma_\theta \cap W_r} |f'| ds + \sum_{C \cap \gamma_\theta \neq \emptyset} \operatorname{diam}(f(C))$$



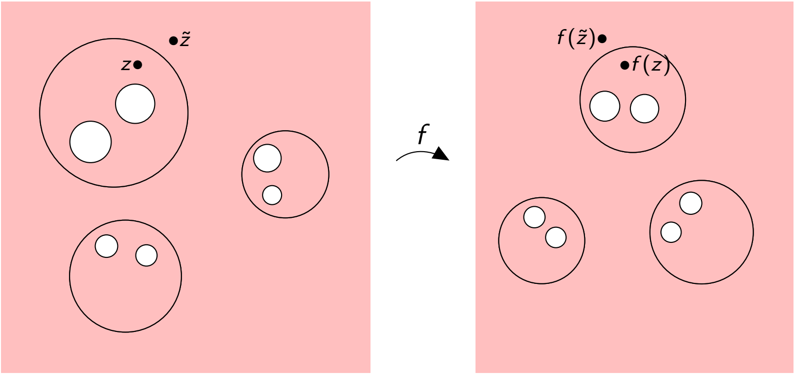
$$r \leq \int_{\gamma_\theta \cap W_r} |f'|ds + \sum_{C \cap \gamma_\theta \neq \emptyset} \operatorname{diam}(f(C)) + \sum_{i=1}^{N} \int_{\gamma_i} |f'|ds$$

The detours γ_i

- γ_i is a concatenation of quasihyperbolic geodesics from the basepoint x_0
- γ_i lies in $N_\varepsilon(\partial\Omega)$: ε-neighborhood of $\partial\Omega$
- γ_i and γ_j intersect distinct Whitney cubes for $i \neq j$

$$\int_0^{2\pi} \sum_{i=1}^{N} \int_{\gamma_i} |f'| ds \, d\theta \lesssim C(f) \cdot \left(\int_{N_\delta(\partial\Omega)} k(x, x_0)^2 dx \right)^{1/2} \xrightarrow{\varepsilon \to 0} 0$$

Step 2: Extension to $\widehat{\mathbb{C}}$

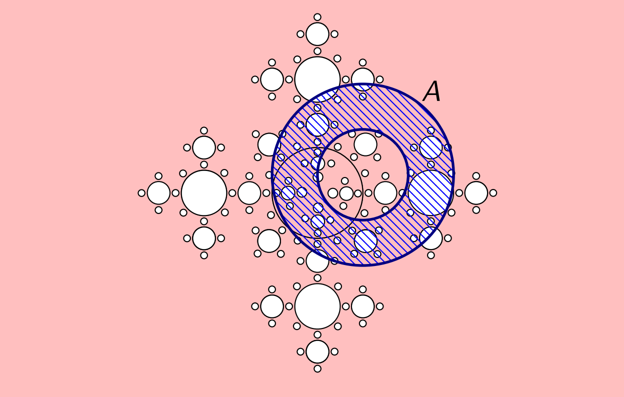


The extension conjugates the **Schottky groups** of Ω and Ω^* and is unique:

$$\left.\begin{array}{l} R = \text{reflection along } C \\ R^* = \text{reflection along } f(C) \end{array}\right\} \quad R^* = f \circ R \circ f^{-1}$$

Step 3: Quasiconformality of f

It suffices to show that $\text{mod}(A) \geq 1$ implies $\text{mod}(f(A)) \geq C$.



- Reflect finitely many times and get $\Omega_k =$ union of **countably many** copies of Ω
- $\text{diam}(A) \gg \text{diam}(C)$, for all circles C in $\partial\overline{\Omega_k}$

We need estimates of the form

$$\mathrm{diam}(f(\gamma)) \leq \int_{\gamma\cap\Omega_k} |f'| ds + \sum_{C\cap\gamma\neq\emptyset} \mathrm{diam}(f(C)) + \text{Error(detours)}$$

- Each reflected copy $T(\Omega)$ of Ω satisfies quasihyperbolic condition (bi-Lipschitz invariant)
- Problem: need **infinitely many** detours!
- Solution: Quasihyperbolic condition in $T(\Omega)$ + continuity $\Rightarrow f$ is **ACL up to** $\partial T(\Omega)$

Step 4: Conformality of f

- $f\colon \Omega \to \Omega^*$ extends uniquely to a K-quasiconformal map of $\widehat{\mathbb{C}}$ that conjugates the Schottky groups of Ω and Ω^*. ($K = K(\Omega)$)
- If $K > 1$, set $\nu = c\mu_f$, $c > 1$. Solve Beltrami equation

$$h_{\overline{z}} = \nu h_z$$

  and get $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that is **not K-quasiconformal**.
- $\mu_h = \nu$ is invariant under the Schottky group of Ω, so $h(\Omega)$ is a **circle domain** and h conjugates the Schottky groups of Ω and $h(\Omega)$.
- $h|_\Omega$ is conformal ($\nu = 0$) so there exists a unique extension $\widetilde{h}$ that is K-**quasiconformal** and conjugates the Schottky groups of Ω and $h(\Omega)$.
- Uniqueness: $\widetilde{h} = h$. Contradiction!

Thank you!