The closed span of an exponential system in L^p spaces on simple closed rectifiable curves in the complex plane and Pólya singularity results for Taylor-Dirichlet series

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New Developments in Complex Analysis and Function Theory
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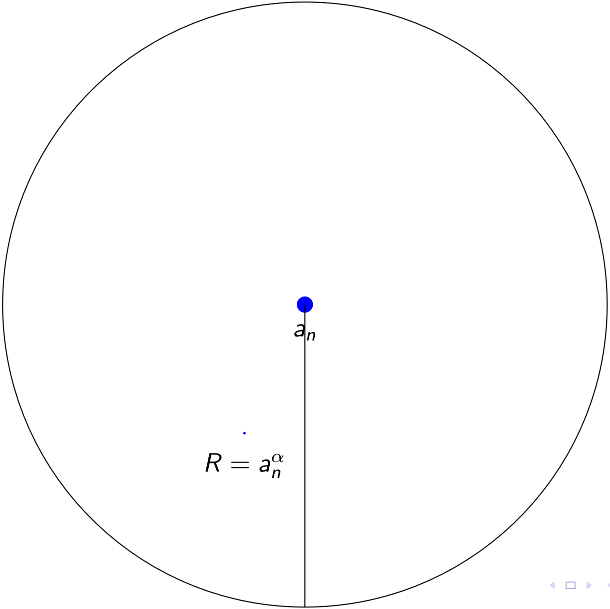

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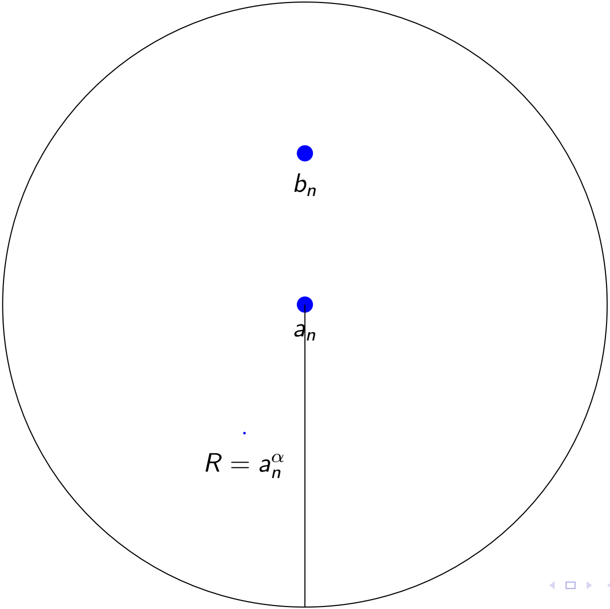

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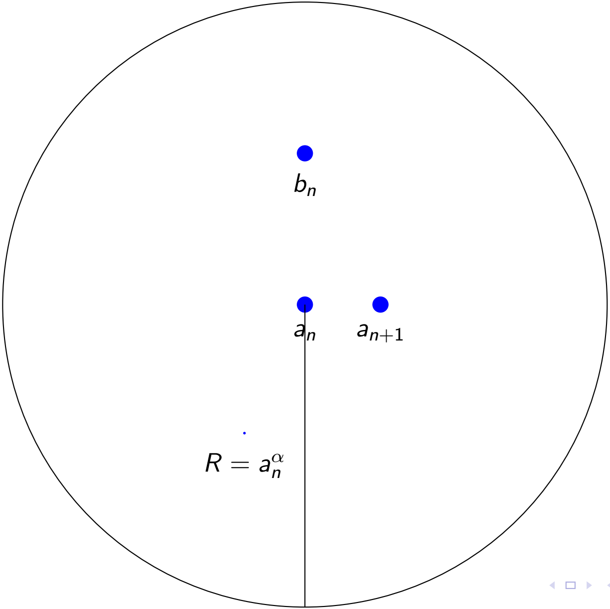

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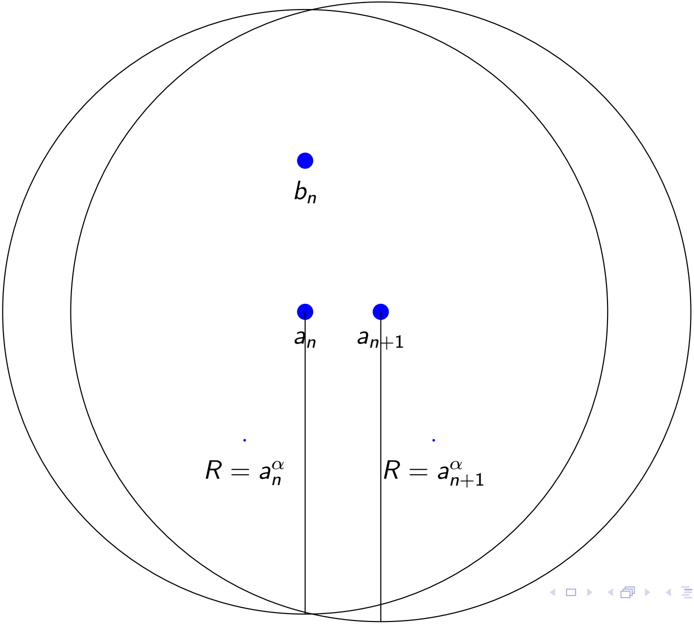

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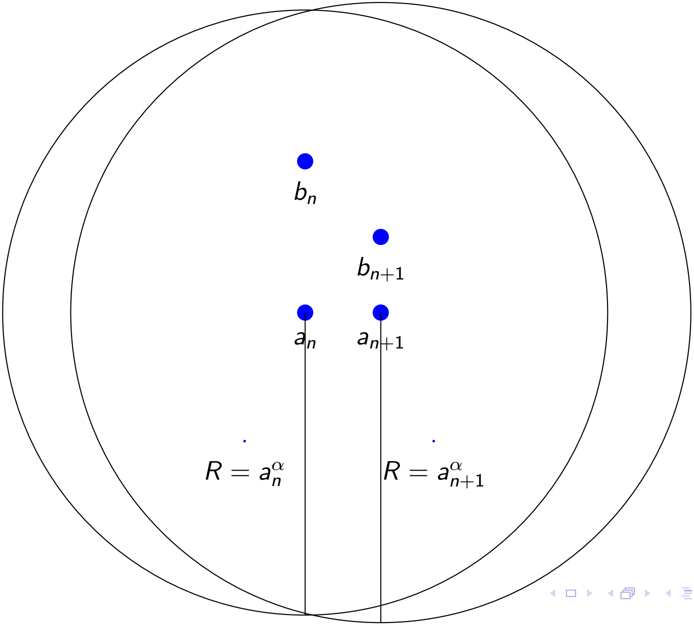

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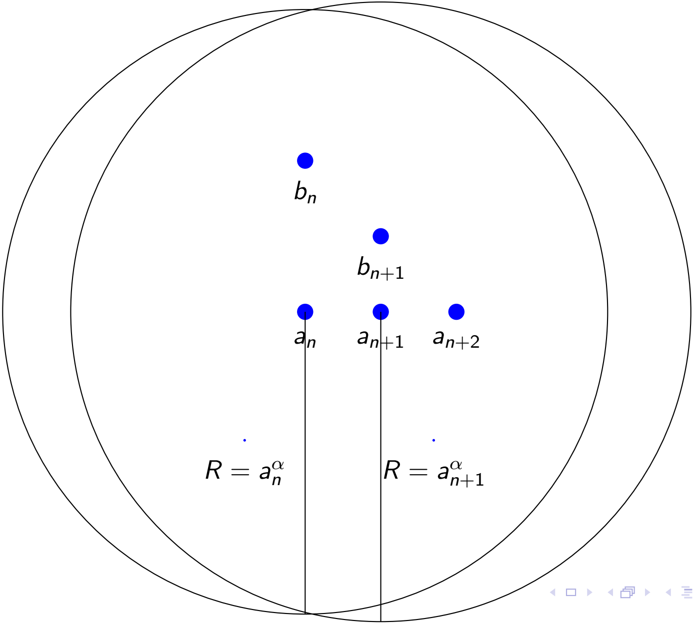

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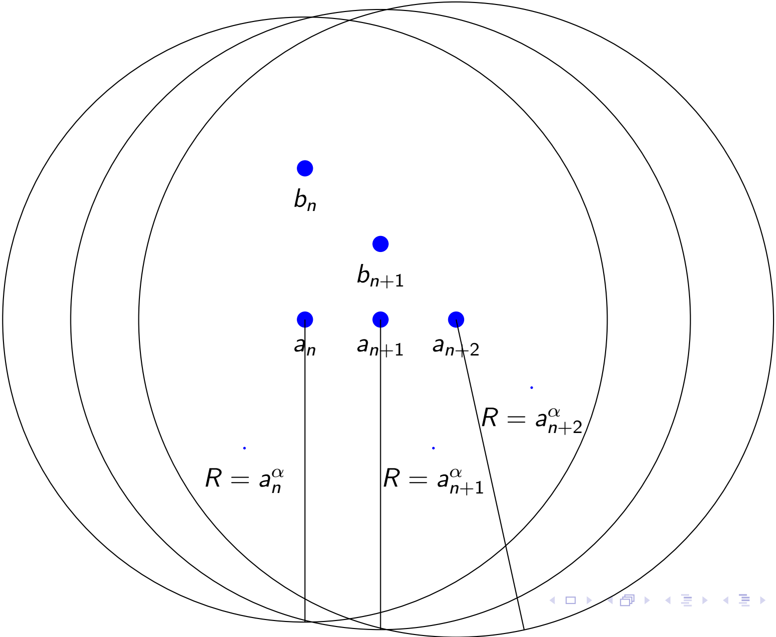

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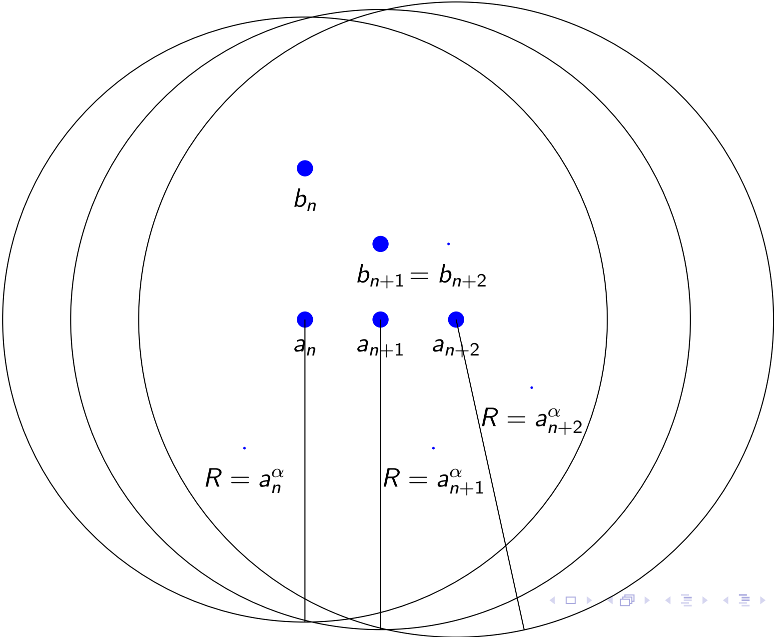

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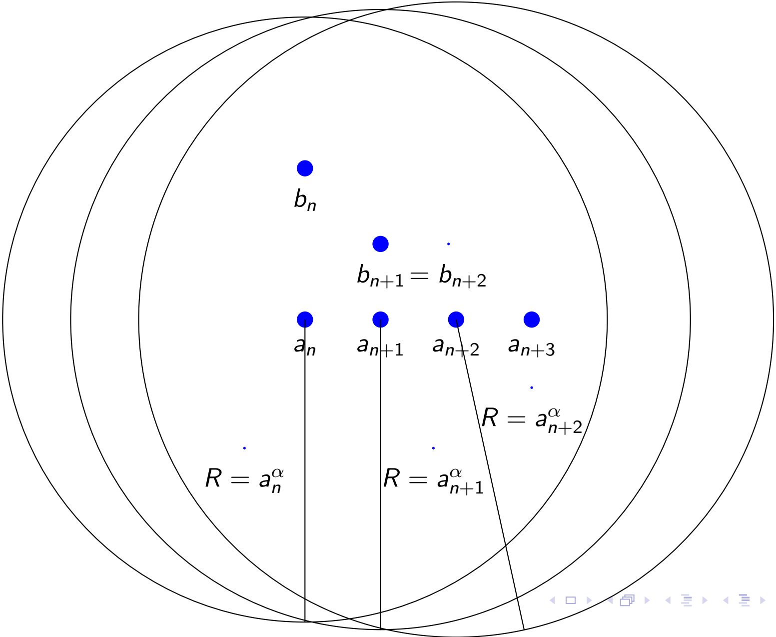

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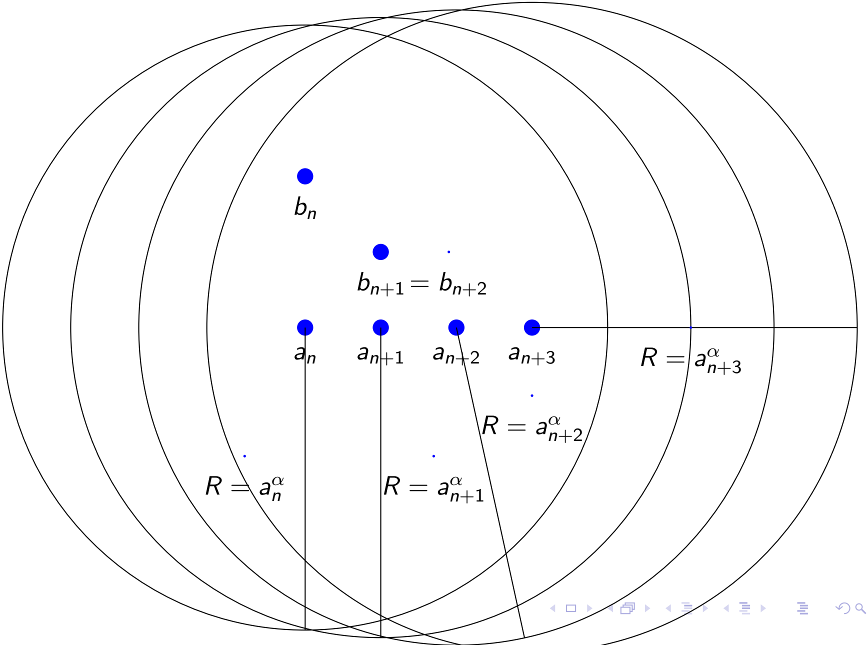

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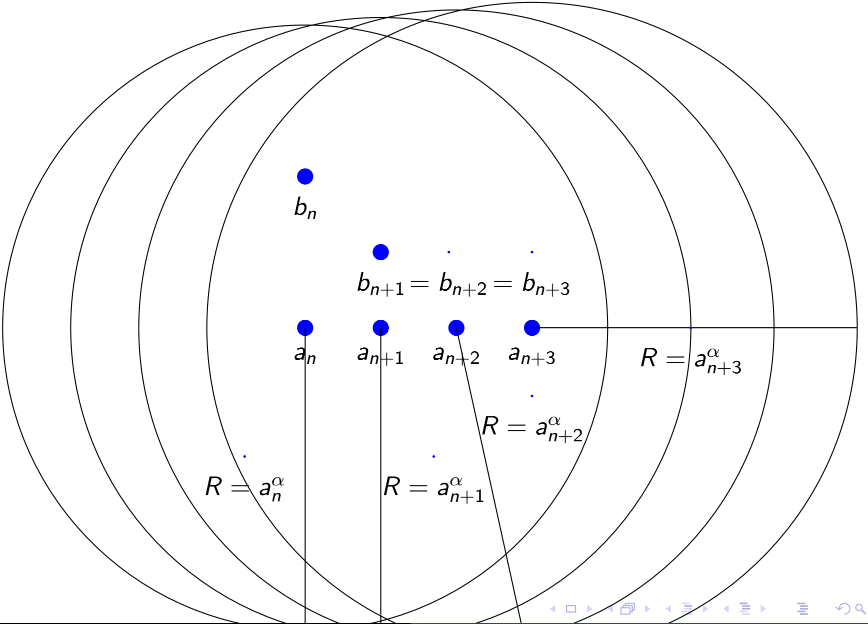

Theorem A

Let the multiplicity-sequence $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ *belong to the class* $U(d, 0)$ *for some* $d > 0$*, and consider the Taylor-Dirichlet series*

$$g(z) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\mu_n - 1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}$$

$$\limsup_{n\to\infty} \frac{\log C_n}{\lambda_n} = \xi \in \mathbb{R}, \qquad \textit{where} \qquad C_n = \max\{|c_{n,k}| : \ k = 0, 1, \ldots, \mu_n - 1\}.$$

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Then $g(z)$ *defines an analytic function in the half-plane* $\{z : \Re z < -\xi\}$ *and it has at least One singularity in every open interval of length exceeding* $2\pi d$ *and lying on the line* $\Re z = -\xi$*.*

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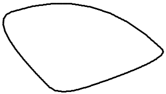

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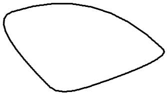

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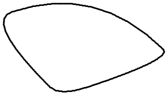

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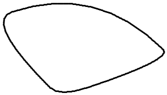

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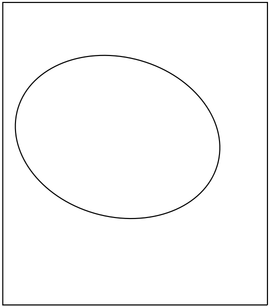

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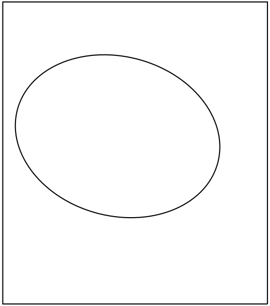

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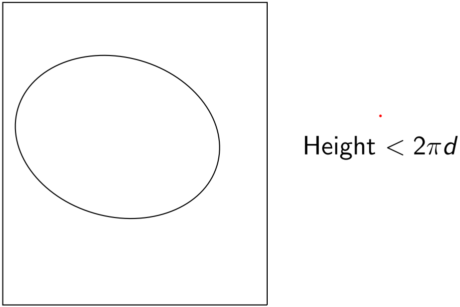

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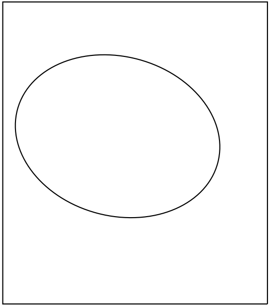

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Hence polynomials are approximated uniformly on the curve l by exponential polynomials.

The curve l is Surrounding a rectangle whose height is $2\pi d$

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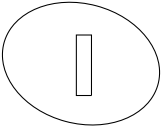

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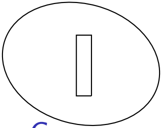

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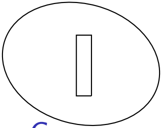

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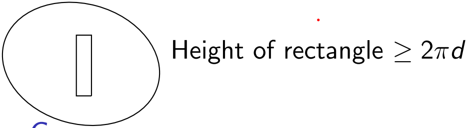

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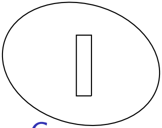

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Question:

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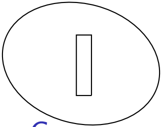

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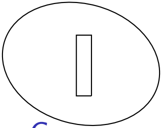
Question: Can we characterize the closed span of the exponential system E_Λ in the space $L^p(l)$ for $p \geq 1$?



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We give an answer when $\Lambda \in U(d, 0)$.

Characterizing the closed span of E_Λ

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Let Λ belong to the class $U(d, 0)$.
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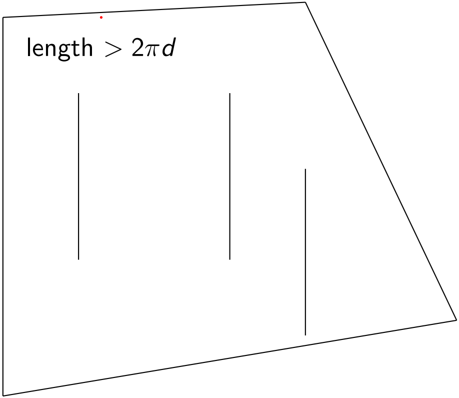
Curve l_d, Domain G_{l_d}



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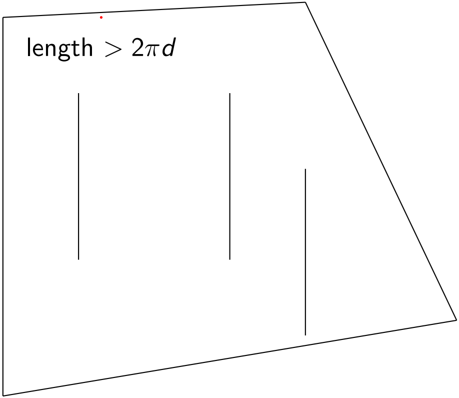
Curve l_d, Domain G_{l_d} S_{l_d} the set of all such line segments



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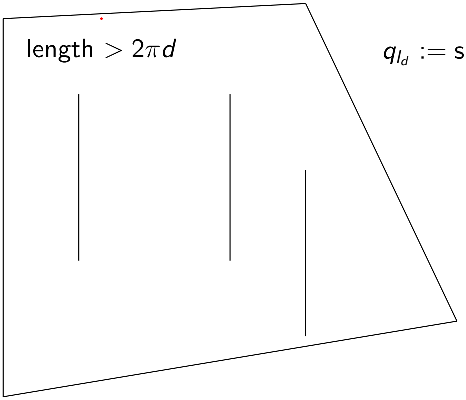

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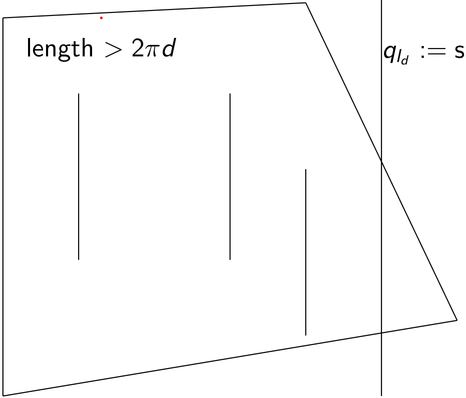

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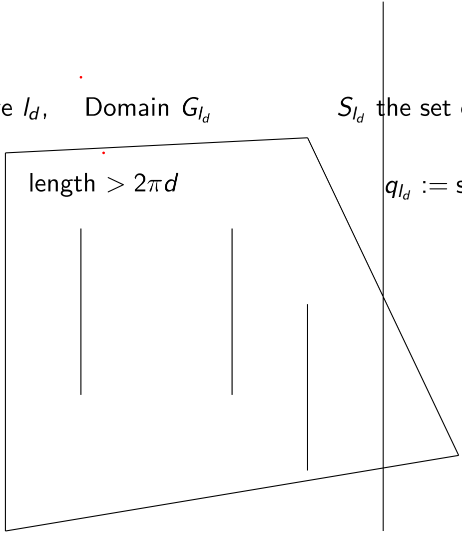

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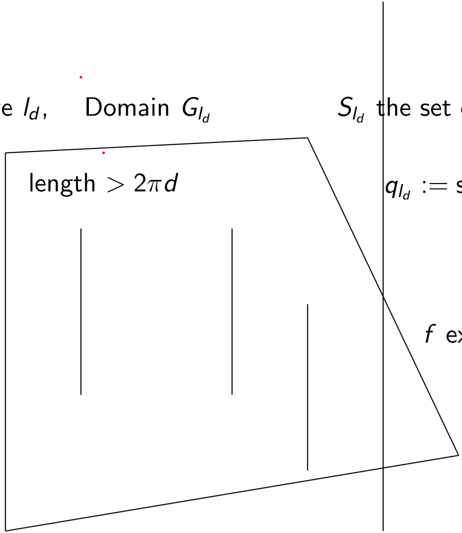

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f extends analytically in $\Re z < q_{l_d}$

as a

Taylor-Dirichlet series

The closed span of E_Λ in $L^p(I_d)$

The closed span of E_Λ in $L^p(l_d)$

Theorem D

Let $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^\infty \in U(d, 0)$ and consider an l_d curve and its q_{l_d} constant.

- *Then every function f belonging to the closed span of E_Λ in $L^p(l_d)$ for $p \geq 1$, not only extends analytically in the domain G_{l_d} and belongs to the Smirnov space $E^p(G_{l_d})$.*
- *But it is also extended analytically in the half-plane $H_{q_{l_d}} := \{z : \Re z < q_{l_d}\}$, admitting a unique Taylor-Dirichlet series representation of the form*

$$g(z) = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\mu_n - 1} c_{n,k} z^k \right) e^{\lambda_n z}, \quad c_{n,k} \in \mathbb{C}, \quad \forall\ z \in H_{q_{l_d}}$$

with the series converging uniformly on compact subsets of $H_{q_{l_d}}$.

Crucial Tool: Distances in $L^p(I_d)$

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Define the **Distance** between $p_{n,k}$ and the closed span of $E_{\Lambda_{n,k}}$ in $L^p(l_d)$

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Theorem E

For every $\epsilon > 0$ there is a constant $u_\epsilon > 0$, independent of $p \geq 1$, $n \in \mathbb{N}$ and $k = 0, 1, \ldots, \mu_n - 1$, but depending on Λ the curve l_d, so that

$$D_{p,n,k} \geq u_\epsilon e^{(q_{l_d} - \epsilon)\lambda_n}.$$

A Biorthogonal sequence to E_Λ in $E^2(G_{l_d})$ and a solution to a Moment Problem

Theorem F

- *Let* $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ *belong to the class* $U(d, 0)$ *and consider an* l_d *curve and its* q_{l_d} *constant.*

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Theorem F

- *Let* $\Lambda = \{\lambda_n, \mu_n\}_{n=1}^{\infty}$ *belong to the class* $U(d, 0)$ *and consider an* l_d *curve and its* q_{l_d} *constant. Then there exists a family of functions*

$$\{r_{n,k} \in E^2(G_{l_d}) : \; n \in \mathbb{N}, \; k = 0, 1, \ldots, \mu_n - 1\}$$

such that this family is the Unique Biorthogonal sequence to the system E_Λ *in* $E^2(G_{l_d})$*, belonging to* $\overline{\mathrm{span}}(E_\Lambda)$ *in* $E^2(G_{l_d})$*.*

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- *Moreover, for every* $\epsilon > 0$ *there is a constant* $m_\epsilon > 0$*, independent of* n *and* k*, but depending on* Λ *and the curve* l_d*, so that*

$$||r_{n,k}||_{E^2(G_{l_d})} \leq m_\epsilon e^{(-q_{l_d}+\epsilon)\lambda_n}, \qquad \forall \; n \in \mathbb{N}, \quad k = 0, 1, \ldots, \mu_n - 1.$$

- Let $\{d_{n,k} : n \in \mathbb{N}, k = 0, 1, \ldots, \mu_n - 1\}$ be a doubly-indexed sequence of complex numbers such that

$$\limsup_{n\to\infty} \frac{\log A_n}{\lambda_n} < q_{l_d} \quad \text{where} \quad A_n = \max\{|d_{n,k}| : k = 0, 1, \ldots, \mu_n - 1\}.$$

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belongs to $E^2(G_{l_d})$ and it is a solution to the moment problem

$$\int_{l_d} \overline{z^k e^{\lambda_n z}} f(z) \, |dz| = d_{n,k} \qquad \forall\ n \in \mathbb{N} \quad \text{and} \quad k = 0, 1, 2, \dots \mu_n - 1.$$

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THANK YOU VERY MUCH!!!

ΣΑΣ ΕΥΧΑΡΙΣΤΩ ΠΑΡΑ ΠΟΛΥ !!!