Removability, rigidity of circle domains and Koebe's conjecture.

Malik Younsi (University of Hawaii)

July 5, 2018

Circle domains

Circle domains

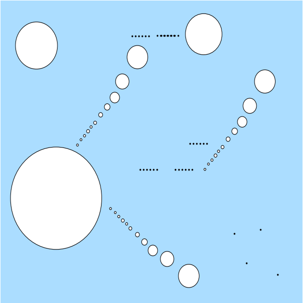
Definition

A domain Ω *in* $\widehat{\mathbb{C}}$ *is a* **circle domain** *if every connected component of its boundary is either a round circle or a point.*

Circle domains

Definition

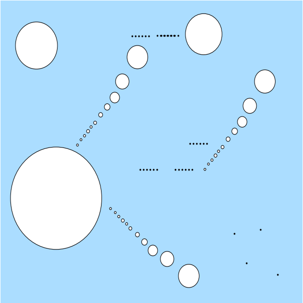
A domain Ω in $\widehat{\mathbb{C}}$ is a **circle domain** *if every connected component of its boundary is either a round circle or a point.*



Circle domains

Definition

A domain Ω in $\widehat{\mathbb{C}}$ is a **circle domain** *if every connected component of its boundary is either a round circle or a point.*



- The boundary of any circle domain contains at most countably many circles.

Koebe's Kreisnormierungsproblem

Koebe's Kreisnormierungsproblem

Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

Koebe's Kreisnormierungsproblem

Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The conjecture is true if

- D has finitely many boundary components (Koebe, 1918)

Koebe's Kreisnormierungsproblem

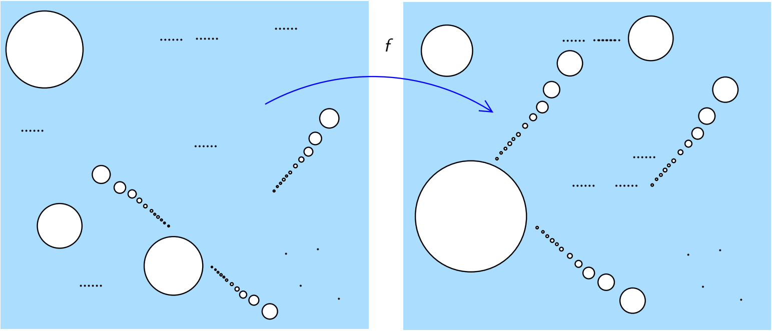
Conjecture (Koebe, 1909)

Every domain $D \subset \widehat{\mathbb{C}}$ is conformally equivalent to a circle domain.

The conjecture is true if

- D has finitely many boundary components (Koebe, 1918)
- D has at most countably many boundary components (He–Schramm, 1993).

Uniqueness of the Koebe map



Conformal rigidity

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ *is* **conformally rigid** *if every conformal map of* Ω *onto another circle domain is the restriction of a Möbius transformation.*

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918).

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918).
- it has countably many boundary components (He–Schramm, 1993).

Conformal rigidity

Definition

A circle domain $\Omega \subset \widehat{\mathbb{C}}$ is **conformally rigid** *if every conformal map of Ω onto another circle domain is the restriction of a Möbius transformation.*

Ω is conformally rigid if

- it has finitely many boundary components (Koebe, 1918).
- it has countably many boundary components (He–Schramm, 1993).

Non-rigid circle domains?

Conformal removability

Conformally removable sets

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Definition

We say that E is **conformally removable** *if every homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Definition

We say that E is **conformally removable** *if every homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Single points, circles, line segments (Morera).

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Definition

We say that E is **conformally removable** *if every homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Single points, circles, line segments (Morera).
- Sets of σ-finite length (Besicovitch, 1931).

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Definition

We say that E is **conformally removable** *if every homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Single points, circles, line segments (Morera).
- Sets of σ-finite length (Besicovitch, 1931).
- Quasicircles.

Conformally removable sets

Let $E \subset \mathbb{C}$ be compact.

Definition

We say that E is **conformally removable** *if every homeomorphism of the Riemann sphere $\widehat{\mathbb{C}}$ which is conformal outside E is actually conformal everywhere (hence is a Möbius transformation).*

- Single points, circles, line segments (Morera).
- Sets of σ-finite length (Besicovitch, 1931).
- Quasicircles.
- The complement of a non-removable Cantor set is a non-rigid circle domain.

Some non-removable sets

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Proof.

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Proof.

Set $\mu := 1/2\chi_E$.

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Proof.

Set $\mu := 1/2\chi_E$. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be quasiconformal with $\mu_f = \mu$.

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Proof.

Set $\mu := 1/2\chi_E$. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be quasiconformal with $\mu_f = \mu$. Then f is a non-Möbius homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E. □

Some non-removable sets

Proposition

Let $E \subset \mathbb{C}$ be a compact set with positive area. Then E is not conformally removable.

Proof.

Set $\mu := 1/2\chi_E$. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be quasiconformal with $\mu_f = \mu$. Then f is a non-Möbius homeomorphism of $\widehat{\mathbb{C}}$ which is conformal outside E. □

- There exist non-removable sets of Hausdorff dimension one and removable sets of Hausdorff dimension two.

A key observation

Going back to rigid circle domains...

A key observation

Going back to rigid circle domains...

Proposition

The complement of a non-removable Cantor set is a non-rigid circle domain.

A key observation

Going back to rigid circle domains...

Proposition

The complement of a non-removable Cantor set is a non-rigid circle domain.

In particular, if E is a Cantor set with $m(E) > 0$, then $\Omega := \widehat{\mathbb{C}} \setminus E$ is non-rigid.

The rigidity conjecture

The rigidity conjecture

Conjecture (He–Schramm, 1994)

Let Ω be a circle domain. The following are equivalent :

(A) *Ω is conformally rigid*

(B) *$\partial\Omega$ is conformally removable*

The rigidity conjecture

Conjecture (He–Schramm, 1994)

Let Ω be a circle domain. The following are equivalent :

(A) Ω is conformally rigid

(B) $\partial\Omega$ is conformally removable

- If there are no circles in $\partial\Omega$, then **(A)** $\Rightarrow$ **(B)**.

Known cases

Known cases

	$\partial\Omega$ **removable?**	Ω **rigid?**
finite	y	y (Koebe 1918)
countable	y	y (He–Schramm 1993)
σ-**finite**	y (Besicovitch 1931)	y (He–Schramm 1994)
John	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Hölder	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Quasi	y (Jones–Smirnov 2000)	y (Ntalampekos-Y. 2018)
Area > 0	NO	NO (Sibner 1968)

The quasihyperbolic condition

The quasihyperbolic condition

- $D \subsetneq \mathbb{C}$ a domain, $\delta_D(x) := \text{dist}(x, \partial D) \qquad (x \in D)$

The quasihyperbolic condition

- $D \subsetneq \mathbb{C}$ a domain, $\delta_D(x) := \mathrm{dist}(x, \partial D) \qquad (x \in D)$

Definition

We define the **quasihyperbolic distance** *of two points* $x_1, x_2 \in D$ *by*

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{1}{\delta_D(x)}\, ds,$$

over all rectifiable paths $\gamma \subset D$ *that connect* x_1 *and* x_2.

The quasihyperbolic condition

- $D \subsetneq \mathbb{C}$ a domain, $\delta_D(x) := \mathrm{dist}(x, \partial D) \qquad (x \in D)$

Definition

We define the **quasihyperbolic distance** *of two points $x_1, x_2 \in D$ by*

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{1}{\delta_D(x)}\, ds,$$

over all rectifiable paths $\gamma \subset D$ that connect x_1 and x_2.

Definition

We say that D satisfies the **quasihyperbolic condition** *if $\int_D k(x, x_0)^2 dx < \infty$ for some $x_0 \in D$.*

The quasihyperbolic condition

- $D \subsetneq \mathbb{C}$ a domain, $\delta_D(x) := \operatorname{dist}(x, \partial D) \qquad (x \in D)$

Definition

We define the **quasihyperbolic distance** *of two points $x_1, x_2 \in D$ by*

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{1}{\delta_D(x)}\, ds,$$

over all rectifiable paths $\gamma \subset D$ that connect x_1 and x_2.

Definition

We say that D satisfies the **quasihyperbolic condition** *if $\int_D k(x, x_0)^2 dx < \infty$ for some $x_0 \in D$.*

- John domains (and more generally Hölder domains) satisfy the quasihyperbolic condition.

How to prove rigidity?

- Let Ω be a circle domain with $\infty \in \Omega$.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0, R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0, R)$.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0, R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0, R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0, R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0, R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Show that f maps circle boundary components to circle boundary components.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0, R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0, R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Show that f maps circle boundary components to circle boundary components.
- Show that f maps point boundary components to point boundary components.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0, R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0, R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Show that f maps circle boundary components to circle boundary components.
- Show that f maps point boundary components to point boundary components.
- Show that f extends continuously to $\overline{\Omega}$.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0,R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0,R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Show that f maps circle boundary components to circle boundary components.
- Show that f maps point boundary components to point boundary components.
- Show that f extends continuously to $\overline{\Omega}$.
- Extend f to a homeomorphism $\widetilde{f}$ of $\widehat{\mathbb{C}}$ by repeated Schwarz reflections.

- Let Ω be a circle domain with $\infty \in \Omega$.
- Suppose that $\Omega \cap B(0,R)$ satisfies the quasihyperbolic condition, where $\widehat{\mathbb{C}} \setminus \Omega \subset B(0,R)$.

Let $f : \Omega \to \Omega^*$ be a conformal map onto another circle domain with $f(\infty) = \infty$.

- Show that f maps circle boundary components to circle boundary components.
- Show that f maps point boundary components to point boundary components.
- Show that f extends continuously to $\overline{\Omega}$.
- Extend f to a homeomorphism $\widetilde{f}$ of $\widehat{\mathbb{C}}$ by repeated Schwarz reflections.
- Show that $\widetilde{f}$ is qc on $\widehat{\mathbb{C}}$.

Quasiconformal rigidity

Quasiconformal rigidity

Theorem (Y., 2016)

A circle domain Ω is conformally rigid if and only if it is quasiconformally rigid.

Further remarks on the rigidity conjecture

Question

If $E \subset \mathbb{C}$ is a conformally removable Cantor set, is $\Omega := \widehat{\mathbb{C}} \setminus E$ a conformally rigid circle domain?

Counterexamples must be wild!

Counterexamples must be wild!

Suppose that E is a removable Cantor set but $\Omega = \widehat{\mathbb{C}} \setminus E$ is not rigid.

Counterexamples must be wild!

Suppose that E is a removable Cantor set but $\Omega = \widehat{\mathbb{C}} \setminus E$ is not rigid.

- There exists a non-Möbius conformal map $f : \Omega \to \Omega^*$, where Ω^* is a circle domain.

Counterexamples must be wild!

Suppose that E is a removable Cantor set but $\Omega = \widehat{\mathbb{C}} \setminus E$ is not rigid.

- There exists a non-Möbius conformal map $f : \Omega \to \Omega^*$, where Ω^* is a circle domain.
- $\partial\Omega^*$ must contain at least one circle.

Counterexamples must be wild!

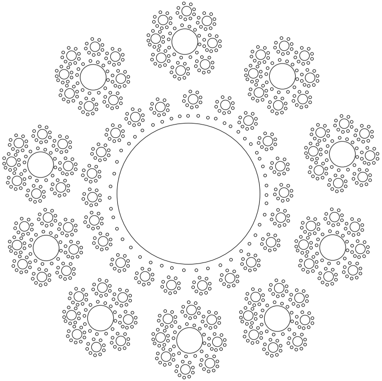
Suppose that E is a removable Cantor set but $\Omega = \widehat{\mathbb{C}} \setminus E$ is not rigid.

- There exists a non-Möbius conformal map $f : \Omega \to \Omega^*$, where Ω^* is a circle domain.
- $\partial\Omega^*$ must contain at least one circle.

Proposition (Ntalampekos–Y. (2018))

Every $w \in \partial\Omega^*$ that is not a point boundary component is the accumulation point of an infinite sequence of distinct circles in $\partial\Omega^*$.

A Sierpinski-type circle domain



Local removability

Local removability

Definition

A compact set $E \subset \mathbb{C}$ is **locally conformally removable** *if for any open set U, every homeomorphism on U that is conformal on $U \setminus E$ is actually conformal on the whole open set U.*

Local removability

Definition

A compact set $E \subset \mathbb{C}$ is **locally conformally removable** *if for any open set U, every homeomorphism on U that is conformal on $U \setminus E$ is actually conformal on the whole open set U.*

- Conformally removable : $E \subset U$.

Local removability

Definition

A compact set $E \subset \mathbb{C}$ is **locally conformally removable** *if for any open set U, every homeomorphism on U that is conformal on $U \setminus E$ is actually conformal on the whole open set U.*

- Conformally removable : $E \subset U$.
- Examples : (compact countable unions of) totally disconnected removable compact sets and quasiarcs.

Local removability

Definition

A compact set $E \subset \mathbb{C}$ is **locally conformally removable** *if for any open set U, every homeomorphism on U that is conformal on $U \setminus E$ is actually conformal on the whole open set U.*

- Conformally removable : $E \subset U$.
- Examples : (compact countable unions of) totally disconnected removable compact sets and quasiarcs.

Conjecture

A compact set is conformally removable if and only if it is locally conformally removable.

Local removability

Definition

A compact set $E \subset \mathbb{C}$ is **locally conformally removable** *if for any open set U, every homeomorphism on U that is conformal on $U \setminus E$ is actually conformal on the whole open set U.*

- Conformally removable : $E \subset U$.
- Examples : (compact countable unions of) totally disconnected removable compact sets and quasiarcs.

Conjecture

A compact set is conformally removable if and only if it is locally conformally removable.

- Would imply that the union of two removable sets is removable.

THANK YOU!