Weak Tangents of Metric Spheres

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Absolute rigor: $\forall R > 0$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$,

$$d_{GH}(\overline{B}_{(X,\lambda_n d)}(x_n, R + \varepsilon), \overline{B}_{(T,d_T)}(x, R)) < \varepsilon.$$

Weak Tangent: Definition

Example

If (X, d) is a compact Riemannian n-manifold, then every weak tangent of (X, d) is $(\mathbb{R}^n, 0)$.

Example

The standard $1/3$-Cantor set has uncountably many weak tangents.

Quasisymmetry

Definition

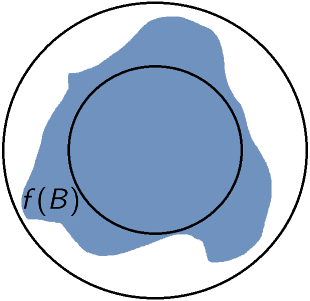
A *quasisymmetry* $\varphi : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is a homeomorphism such that for all $x, y, z \in X$ with $x \neq z$,

$$\frac{d_Y(\varphi(x), \varphi(y))}{d_Y(\varphi(x), \varphi(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right).$$

where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism.

Intuition: there exists $C > 1$ such that for every ball $B \subset X$, there exists another ball $B' \subset Y$ such that

$$B' \subset \varphi(B) \subset CB'.$$



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Remark: If $\varphi : X \to Y$ is a quasisymmetry, then $\varphi^{-1} : Y \to X$ is a quasisymmetry.

Weak Tangents and Quasisymmetries

Lemma

If $\varphi : X \to Y$ *is a quasisymmetry, and if* $(X, x_n, \lambda_n d_X) \to T$, *then for some* μ_n, $(Y, \varphi(x_n), \mu_n d_Y)$ *has a converging subsequence whose limit* T' *is quasisymmetric to* T.

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Theorem (Kinneberg [3])

A doubling metric circle $\mathcal{C}$ is a quasicircle if and only if every weak tangent of $\mathcal{C}$ is quasisymmetric to $(\mathbb{R}, 0)$.

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Theorem (W. [4])

For all $n \geq 2$, there exists a doubling, LLC metric space X homeomorphic to $\mathbb{S}^n$ such that every weak tangent of X is isometric to $(\mathbb{R}^n, 0)$ but X is not quasisymmetric to $\mathbb{S}^n$.

Weak Tangents and Quasisymmetries

Theorem (Bonk, Kleiner [1])

Suppose Z is a uniformly perfect, doubling compact metric space, and $G \curvearrowright Z$ is a uniformly quasi-Möbius action for which the induced action $G \curvearrowright Tri(Z)$ is cocompact. If (T, p) is a weak tangent of Z, then there exists a quasi-Möbius homeomorphism $h : (\widehat{S}, \widehat{d_p}) \to Z$.

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Taking $G =$ finitely generated infinite hyperbolic group, $Z = \partial_\infty G$:

Corollary

Suppose $\partial_\infty G$ is homeomorphic to $\mathbb{S}^n$. If any weak tangent of $\partial_\infty G$ is quasisymmetric to $(\mathbb{R}^n, 0)$, then $\partial_\infty G$ is quasisymmetric to $\mathbb{S}^n$.

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Conjecture (Cannon's conjecture)

If $\partial_\infty G$ is homeomorphic to $\mathbb{S}^2$, then $\partial_\infty G$ is quasisymmetric to $\mathbb{S}^2$.

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1. branched covering : $\forall x \in \mathbb{S}^2, \exists$ open $U \ni x$, $\exists$ homeo φ, ψ,

$$\begin{array}{ccc} (U, x) & \xrightarrow{f} & (f(U), f(x)) \\ \downarrow \varphi & & \downarrow \psi \\ (\mathbb{D}, 0) & \xrightarrow{z \mapsto z^d} & (\mathbb{D}, 0). \end{array}$$

Weak Tangents and Expanding Thurston Maps

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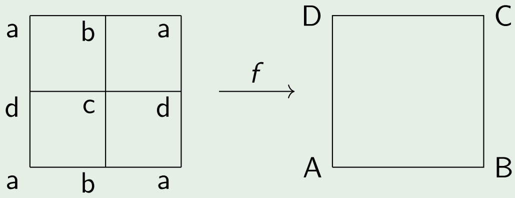
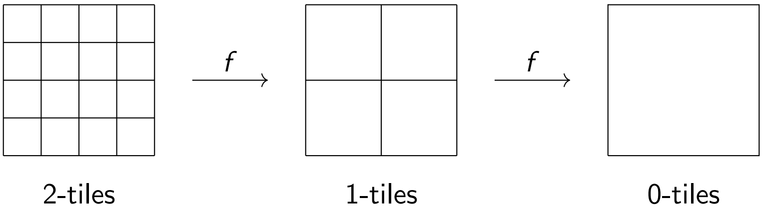
Example (The 2×2-subdivision map)



Visual Spheres of Expanding Thurston Maps



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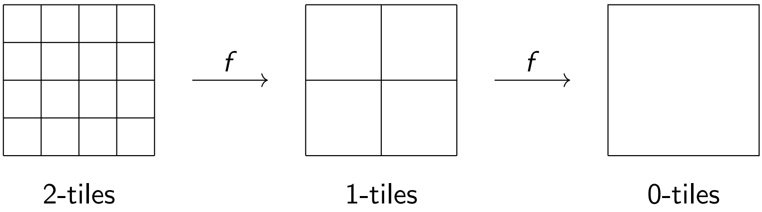

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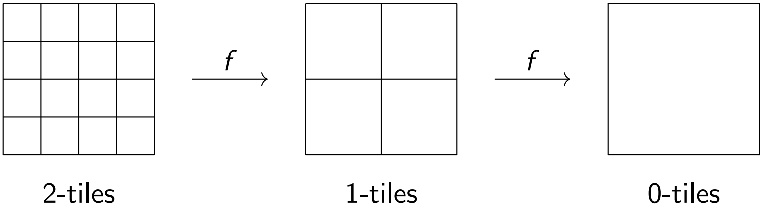

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1. There exists $N \in \mathbb{N}$ and f^N-invariant Jordan curve $\mathcal{C}$ containing all postcritical points [2]. WLOG $N = 1$.

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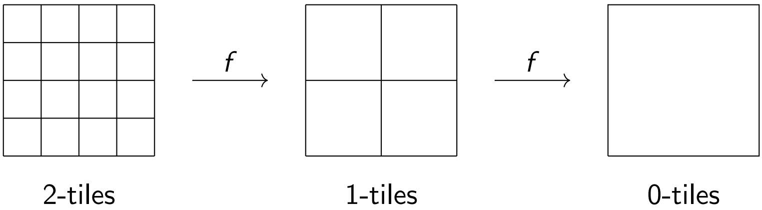

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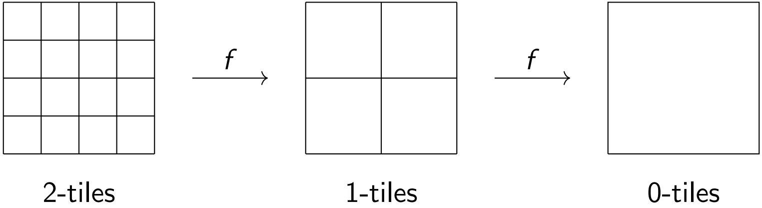

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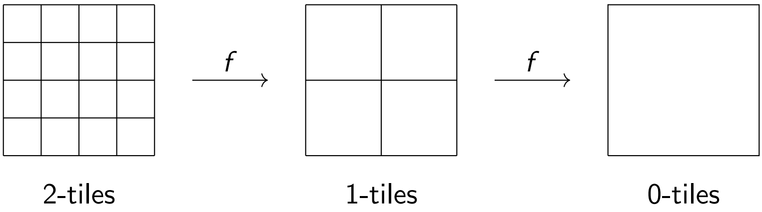

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4. $f^{-n}(\mathcal{C})$ gives us n-tiles.

Visual Spheres of Expanding Thurston Maps

Proposition (Bonk and Meyer [2])

There exists $\Lambda > 1$ and a metric ρ on $\mathbb{S}^2$ that generates the topology of $\mathbb{S}^2$ and such that

$$\operatorname{diam}_\rho(n\text{-}tile) \approx \Lambda^{-n}.$$

- ρ is a *visual metric* with respect to f
- Λ is the *expansion factor* of ρ.
- $(\mathbb{S}^2, \rho)$ is a *visual sphere*.
- visual metric always exists for f but not unique.
- two visual spheres for f are snowflake equivalent, therefore quasisymmetric.

Theorem

Let f be an expanding Thurston map with no periodic critical point and ρ be a visual metric with respect to f. TFAE:

(i) *f is Thurston equivalent to a rational map.*

(ii) *$(\mathbb{S}^2, \rho)$ is a quasisphere.*

(iii) *Every weak tangent of $(\mathbb{S}^2, \rho)$ is quasisymmetric to $(\mathbb{R}^2, 0)$.*

(iv) *Some weak tangent of $(\mathbb{S}^2, \rho)$ is quasisymmetric to $(\mathbb{R}^2, 0)$.*

(i) $\iff$ (ii) Bonk and Meyer [2].
(ii) $\implies$ (iii) Lemma.
(iii) $\implies$ (iv) Clear since $(\mathbb{S}^2, \rho)$ has a weak tangent.
(iv) $\implies$ (ii) W.

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