Closed range composition operators on *BMOA*

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Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$
- $\varphi : \mathbb{D} \to \mathbb{D}$ analytic self map of $\mathbb{D}$
- $\alpha_q(z) = \frac{q-z}{1-\overline{q}z}$, $q \in \mathbb{D}$ Mobius transformation
- $D(q, r) = \{z \in \mathbb{D} : |\alpha_q(z)| < r\}$, $r \in (0, 1)$
- $H(\mathbb{D})$ is the set of analytic functions on $\mathbb{D}$

Composition operators

- $C_\varphi f = f \circ \varphi$
- C_φ is a linear operator
- Let φ be a non constant self map of $\mathbb{D}$
  By the Open Mapping Theorem for analytic functions, $\varphi(\mathbb{D})$ is an open subset of $\mathbb{D}$
- $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is one to one

The *Hardy space* H^2

- H^2 is the Hilbert space of analytic functions f on $\mathbb{D}$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \ ||f||_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

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- an equivalent norm on H^2 is given by

$$||f||_{H^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} (1 - |z|^2)|f'(z)|^2 dA(z)$$

BMOA

- *BMOA* is the Banach space of analytic functions on $\mathbb{D}$

$$||f||_G = \sup_{q\in\mathbb{D}} ||f \circ \alpha_q - f(q)||_{H^2} < \infty$$

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- The norm we will use in *BMOA* is:

$$||f||_{BMOA} = |f(0)| + ||f||_*$$

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$$\begin{aligned} \|f \circ \varphi\|_*^2 &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \\ &= \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 (1 - |\alpha_q(z)|^2) dA(z) \end{aligned}$$

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Counting functions on *BMOA*

For each $q \in \mathbb{D}$

- we define the *BMOA* counting function by

$$N_{q,\varphi}(\zeta) = \sum_{\varphi(z)=\zeta} (1 - |\alpha_q(z)|^2)$$

  if $\zeta \notin \varphi(\mathbb{D})$, $N_{q,\varphi}(\zeta) = 0$

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$$\begin{aligned} ||C_\varphi f||_*^2 &= \sup_{q\in\mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta)\, dA(\zeta) \\ &\leq \text{const.}\, ||f||_*^2 \end{aligned}$$

C_φ bounded below, closed range on $BMOA$

- C_φ is bounded below on $BMOA \Leftrightarrow \exists C > 0$ such that $\forall f \in BMOA$

$$||f||_{BMOA} \leq C\,||f \circ \varphi||_{BMOA}$$

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$$||f \circ \varphi||_* \asymp ||f||_*$$

- We say that C_φ is closed range in $BMOA$ if $C_\varphi(BMOA)$ is closed in $BMOA$
- C_φ is closed range $\Leftrightarrow$ C_φ is bounded below

- $C_\varphi : H^2 \to H^2$

  J. Cima, J. Thomson, W. Wogen (1973)

  $\nu(E) := m(\varphi^{-1}(E))$, for E any Borel subset of $\mathbb{T}$

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- N. Zorboska (1994), K. Luery (2013) and P. Ghatage, MT (2014)

- We define $H \subseteq \mathbb{D}$ to be a sampling set for *BMOA* if for all $f \in BMOA$

$$\sup_{q \in \mathbb{D}} \int_H |f'(z)|^2 \left(1 - |\alpha_q(z)|^2\right) dA(z) \asymp ||f||_*^2 .$$

Sampling sets in *BMOA*

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- For each $\varepsilon > 0$ and $q \in \mathbb{D}$

$$G_{\varepsilon,q} = \{\zeta \;:\; N_{q,\varphi}(\zeta) > \varepsilon\left(1 - |\alpha_q(\zeta)|^2\right)\}$$

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- C_φ is closed range on *BMOA* $\Rightarrow \exists \, \varepsilon > 0$ such that $\cup_{q\in\mathbb{D}} G_{\varepsilon,q}$ is a sampling set for *BMOA*
- $\cap_{q\in\mathbb{D}} G_{\varepsilon,q}$ is a sampling set for *BMOA* $\Rightarrow C_\varphi$ is closed range on *BMOA*

Carleson measures

- Let μ be a finite positive Borel measure on $\mathbb{D}$. We say that μ is a (Bergman space) Carleson measure on $\mathbb{D}$ if there exists $c > 0$ such that for all $f \in A^2$

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^2 \, dA(z)$$

- Let $0 < r < 1$. Then μ is a Carleson measure if and only if there exists $c_r > 0$ such that for all $q \in \mathbb{D}$,

$$\mu(D(q,r)) \leq c_r \, A(D(q,r))$$

- The Berezin symbol of μ is

$$\tilde{\mu}(q) = \int_{\mathbb{D}} |\alpha_q'(z)|^2 \, d\mu(z) \,, \quad q \in \mathbb{D} \,.$$

- μ is Carleson measure if and only if $\tilde{\mu}$ is a bounded function on $\mathbb{D}$

- The Berezin symbol of μ is

$$\tilde{\mu}(q) = \int_{\mathbb{D}} |\alpha_q'(z)|^2 \, d\mu(z) \, , \quad q \in \mathbb{D} \, .$$

- μ is Carleson measure if and only if $\tilde{\mu}$ is a bounded function on $\mathbb{D}$
- $\mu_{q'}$, $q' \in \mathbb{D}$, is a collection of uniformly Carleson measures if and only if the Berezin symbols of $\mu_{q'}$, for all $q' \in \mathbb{D}$, are uniformly bounded in $\mathbb{D}$.
- Recall

$$||C_\varphi f||_*^2 = \sup_{q \in \mathbb{D}} \int_{\mathbb{D}} |f'(\zeta)|^2 N_{q,\varphi}(\zeta) \, dA(\zeta) \leq \text{const.} \, ||f||_*^2$$

- The measures $N_{q,\varphi}(\zeta) \, dA(\zeta)$ are uniformly Carleson measures

Dan Luecking and the RCC

- Let μ be a finite positive Carleson measure on $\mathbb{D}$. We say that μ satisfies the reverse Carleson condition if $\exists\ r \in (0,1)$ such that

$$\mu(D(q,r)) \asymp A(D(q,r))\,, \quad q \in \mathbb{D}$$

- $G \subset \mathbb{D}$ satisfies the reverse Carleson condition if the Carleson measure $\chi_G(z)\, dA(z)$ satisfies the reverse Carleson condition. Luecking $\Leftrightarrow$

$$\int_{\mathbb{D}} |f(z)|^2\, dA(z) \leq C \int_G |f(z)|^2\, dA(z)\,, \qquad \forall f \in A^2$$

$\Leftrightarrow A(G \cap D(q,r)) \asymp A(D(q,r)),\ q \in \mathbb{D}$

- A subset H of $\mathbb{D}$ satisfies the reverse Carleson condition if and only if H is a sampling set for $BMOA$.

$G_{\varepsilon,q',q}$

- A subset H of $\mathbb{D}$ satisfies the reverse Carleson condition if and only if H is a sampling set for $BMOA$.
- For each $\varepsilon > 0$ and $q, q' \in \mathbb{D}$

$$G_{\varepsilon,q',q} = \{\zeta \ : \ N_{q',\varphi}(\zeta) > \varepsilon\,(1 - |\alpha_q(\zeta)|^2)\}$$

- A subset H of $\mathbb{D}$ satisfies the reverse Carleson condition if and only if H is a sampling set for $BMOA$.
- For each $\varepsilon > 0$ and $q, q' \in \mathbb{D}$

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- $\exists\ k > 0\ \forall\ q \in \mathbb{D}\ ||\alpha_q \circ \varphi||_* \geq k \Leftrightarrow \exists\ \varepsilon > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D}$, $\exists\ q' \in \mathbb{D}$ such that

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1\,. \tag{1}$$

Given a measurable set F, the following are equivalent:

- $\exists\ \delta > 0,\ r \in (0,1)$ such that $\forall$ disks D with centers on $\mathbb{T}$,

$$|F \cap D| > \delta\, |\mathbb{D} \cap D|\,.$$

- $\exists\ \delta_0 > 0,\ \eta \in (0,1)$ such that $\forall q \in \mathbb{D}$

$$|F \cap \mathbb{D}\,(q, \eta(1-|q|))\,| > \delta_0\, |\mathbb{D}\,(q, \eta(1-|q|))\,|\,.$$

- $\exists\ \delta_1 > 0,\ r \in (0,1)$ such that $\forall q \in \mathbb{D}$

$$|F \cap D(q,r)| > \delta_1\, |D(q,r)|\,.$$

Given a collection of measurable sets F_q, $q \in \mathbb{D}$, the following are equivalent:

- $\exists\ \delta > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D}$ and $\forall$ disks D with centers on $\mathbb{T}$, $\exists\ q' \in \mathbb{D}$ such that
$$|F_{q'} \cap D| > \delta\, |\mathbb{D} \cap D|\,.$$
- $\exists\ \delta_0 > 0,\ \eta \in (0,1)$ such that $\forall\ q \in \mathbb{D}\ \exists\ q' \in \mathbb{D}$ such that
$$|F_{q'} \cap \mathbb{D}\,(q, \eta(1-|q|))\,| > \delta_0\, |\mathbb{D}\,(q, \eta(1-|q|))\,|\,.$$
- $\exists\ \delta_1 > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D}\ \exists\ q' \in \mathbb{D}$ such that
$$|F_{q'} \cap D(q,r)| > \delta_1\, |D(q,r)|\,.$$

A reminder

- $G \subset \mathbb{D}$ satisfies the reverse Carleson condition if the Carleson measure $\chi_G(z)\, dA(z)$ satisfies the reverse Carleson condition. Luecking $\Leftrightarrow$

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) \leq C \int_G |f(z)|^2 \, dA(z) \,, \qquad \forall f \in A^2$$

$\Leftrightarrow$ $A(G \cap D(q,r)) \asymp A(D(q,r))$, $q \in \mathbb{D}$

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  $\Leftrightarrow A(G \cap D(q,r)) \asymp A(D(q,r)),\ q \in \mathbb{D}$

- $\exists\ k > 0\ \forall\ q \in \mathbb{D}\ ||\alpha_q \circ \varphi||_* \geq k \Leftrightarrow \exists\ \varepsilon > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D},\ \exists\ q' \in \mathbb{D}$ such that

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1\,.$$

- $\Rightarrow C_\varphi\ : BMOA \to BMOA$ is closed range?

- The Bloch space $\mathcal{B}$ is the set of functions f analytic on $\mathbb{D}$ such that

$$||f||_B := |f(0)| + ||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty$$

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- $$||f||_{\mathcal{B}} \asymp \sup_{q\in\mathbb{D}} ||f\circ\alpha_q - f(q)||_{A^2}$$

- $$||f||_{*} \asymp \sup_{q\in\mathbb{D}} ||f\circ\alpha_q - f(q)||_{H^2}$$

- $BMOA \subset \mathcal{B}$

- $C_\varphi : \mathcal{B} \to BMOA$ is closed range $\Leftrightarrow$
$\exists\ \varepsilon > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D}$, $\exists\ q' \in \mathbb{D}$ such that

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- $C_\varphi : \mathcal{B} \to BMOA$ is closed range $\Leftrightarrow \exists\ k > 0$ such that $\forall q \in \mathbb{D}\ ||\alpha_q \circ \varphi||_* \geq k$

$C_\varphi : BMOA \to BMOA$, etc.

- $C_\varphi : BMOA \to BMOA$ is closed range $\Rightarrow \exists\ k > 0\ \forall q \in \mathbb{D}$
  $||\alpha_q \circ \varphi||_* \geq k$

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- $C_\varphi : BMOA \to BMOA$ is closed range $\Leftrightarrow \exists\, k > 0\ \forall q \in \mathbb{D}$ $||\alpha_q \circ \varphi||_* \geq k$
- Assuming that $C_\varphi : X \to X$ is a bounded operator, $C_\varphi : X \to X$ is closed range $\Leftrightarrow \exists\, k > 0\ \forall q \in \mathbb{D}$ $||\alpha_q \circ \varphi||_X \geq k$, where $X = \mathcal{B}$, Besov type spaces, Q_p

- J. Akeroyd, P. Ghatage, M.T:

  C_φ is closed range on $\mathcal{B} \Leftrightarrow ||\alpha_{\boldsymbol{q}} \circ \varphi||_{\mathcal{B}} \asymp 1,\ \boldsymbol{q} \in \mathbb{D}$

$C_\varphi : \mathcal{B} \to \mathcal{B}$ versus $C_\varphi : BMOA \to BMOA$

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- C_φ is closed range on $\mathcal{B} \Rightarrow C_\varphi$ is closed range on $BMOA$.

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- C_φ is closed range on $\mathcal{B} \Rightarrow C_\varphi$ is closed range on $BMOA$.

- P. Ghatage, D. Zheng, N. Zorboska: for φ univalent

  C_φ closed range on $BMOA \Rightarrow C_\varphi$ closed range on $\mathcal{B}$

- We conclude:
  C_φ is closed range on $\mathcal{B} \Leftrightarrow C_\varphi$ is closed range on $BMOA$

- N. Zorboska (1994):

C_φ is closed range on $H^2 \Leftrightarrow \exists\ \varepsilon > 0$ such that the set

$$G_{\varepsilon,0,0} = \{\zeta \in \mathbb{D} : \sum_{\varphi(z)=\zeta} (1 - |z|^2) > \varepsilon(1 - |\zeta|^2)\}$$

satisfies the RCC

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- C_φ is closed range on $H^2 \Rightarrow C_\varphi$ is closed range on $BMOA$

$||\alpha_q \circ \varphi||_* \asymp 1, \; q \in \mathbb{D}$

- J. Laitila (2010)

  isometries among composition operators on *BMOA* using the seminorm $||f||_G$,

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- Below we give another characterization of closed range composition operators on *BMOA*:

  $\exists\ k \in (0,1]$ such that $\forall\ q \in \mathbb{D}$, $||\alpha_q \circ \varphi||_* \geq k \Leftrightarrow$

  $\exists\ k \in (0,1]$ such that $\forall\ q \in \mathbb{D}\ \exists\ q' \in \mathbb{D}$ with $|\alpha_q(q')|^2 \leq 1 - k^2$, $\exists$ a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to q'$ and

$$\lim_{n \to \infty} ||\varphi_{q_n}||_{H^2} \geq k\,,$$

  where $\forall\ n$, $\varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$

Recall

$\forall\ \varepsilon > 0$ and $q, q' \in \mathbb{D}$

$$G_{\varepsilon,q',q} = \{\zeta\ :\ N_{q',\varphi}(\zeta) > \varepsilon\,(1 - |\alpha_q(\zeta)|^2)\}$$

Main theorem: The following statements are equivalent:

(a) $\exists\ k > 0\ \forall\ q \in \mathbb{D}\ ||\alpha_q \circ \varphi||_* \geq k$

(b) $\exists\ \varepsilon > 0,\ r \in (0,1)$ such that $\forall\ q \in \mathbb{D},\ \exists\ q' \in \mathbb{D}$ such that

$$\frac{|G_{\varepsilon,q',q} \cap D(q,r)|}{|D(q,r)|} \asymp 1\,.$$

(c) $C_\varphi\ : \mathcal{B} \to BMOA$ is closed range

(d) $C_\varphi\ : BMOA \to BMOA$ is closed range

(e) $\exists\ k \in (0,1]$ such that $\forall\ q \in \mathbb{D}\ \exists\ q' \in \mathbb{D}$ with $|\alpha_q(q')|^2 \leq 1 - k^2$, $\exists$ a sequence (q_n) in $\mathbb{D}$ such that $\varphi(q_n) \to q'$ and

$$\lim_{n\to\infty} ||\varphi_{q_n}||_2 \geq k$$

where $\forall n,\ \varphi_{q_n} = \alpha_{\varphi(q_n)} \circ \varphi \circ \alpha_{q_n}$

Thank you