Hedgehogs in Higher Dimensions

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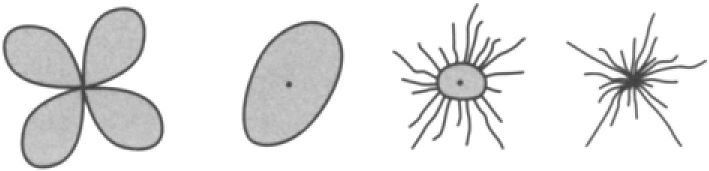
We will examine some results about hedgehog dynamics in $\mathbb{C}$ with the purpose of transporting them to higher dimensions.

Theorem (Pérez-Marco)

Let $f(z) = \lambda z + \mathcal{O}(z^2)$, with $|\lambda| = 1$, be a local holomorphic diffeomorphism. Let U be a Jordan domain containing 0 such that f and f^{-1} are defined and univalent in a neighborhood of $\overline{U}$. There exists a compact, connected set K containing 0, such that $\mathbb{C} \setminus K$ is connected, $f(K) = K$, $f^{-1}(K) = K$ and $K \cap \partial U \neq \emptyset$.

If $f^{\circ n} \neq id\ \forall n \in \mathbb{N}$ then f is linearizable if and only if $0 \in \operatorname{int}(K)$.



Possible types of hedgehogs (by Pérez-Marco).

Pérez-Marco's results

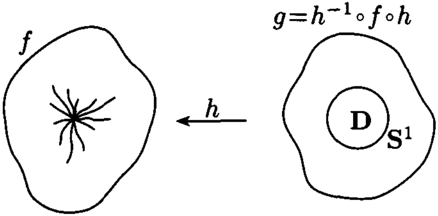
Consider U a C^1-Jordan domain and a germ $f(z) = \lambda z + \mathcal{O}(z^2)$, $\lambda = e^{2\pi i \alpha}$ and $\alpha \notin \mathbb{Q}$.

- The hedgehog is unique and equal to the connected component containing 0 of the set $\{z \in \overline{U} : f^n(z) \in \overline{U} \text{ for all } n \in \mathbb{Z}\}$.

If f is non-linearizable then:

- The hedgehog has empty interior and is not locally connected.
- All points on the hedgehog K are recurrent. The dynamics on the hedgehog has no periodic point except the fixed point at 0.
- If $x \in U \setminus K$ then x cannot converge to a point of K under iterations of f.

Let K be a hedgehog for f and $\lambda = e^{2\pi i\alpha}$. One can associate to K an analytic circle diffeomorphism with rotation number α as follows. Uniformize $\mathbb{C} \setminus K$ using the Riemann map $h : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$. The mapping $g = h^{-1} \circ f \circ h$ is defined and holomorphic in an open annulus $\{1 < |z| < r\}$ and can be extended to the annulus $\{1/r < |z| < r\}$ by the Schwarz reflexion principle. The restriction $g|_{\mathbb{S}^1}$ is a real-analytic diffeomorphism with rotation number α.



Semi-neutral holomorphic germs of $(\mathbb{C}^2, 0)$

Let f be a holomorphic germ of diffeomorphisms of $(\mathbb{C}^2, 0)$. A fixed point x of f is *semi-indifferent* (or *semi-neutral*) if the eigenvalues λ and μ of df_x satisfy $|\lambda| = 1$ and $|\mu| < 1$.

1. semi-parabolic: $\lambda = e^{2\pi i p/q}$, $|\mu| < 1$
2. semi-Siegel: $\lambda = e^{2\pi i \alpha}$, $|\mu| < 1$, where $\alpha \notin \mathbb{Q}$ and there exists an injective holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2$ such that

$$f(\varphi(\xi)) = \varphi(\lambda\xi), \text{ for all } \xi \in \mathbb{D}$$

3. semi-Cremer: $\lambda = e^{2\pi i \alpha}$, $|\mu| < 1$, where $\alpha \notin \mathbb{Q}$ and the fixed point is not semi-Siegel.

Partially hyperbolic germs

The map f is partially hyperbolic on a set B if there exist two real numbers μ_1 and λ_1 such that $0 < |\mu| < \mu_1 < \lambda_1 < 1$ and a family of invariant cone fields $\mathcal{C}^{h/v}$ on B

$$df_x(\mathcal{C}_x^h) \subset \text{ Int } \mathcal{C}_{f(x)}^h \cup \{0\}, \quad df_x^{-1}(\mathcal{C}_{f(x)}^v) \subset \text{ Int } \mathcal{C}_x^v \cup \{0\},$$

such that for some Riemannian metric we have strong contraction in the vertical cones, whereas in the horizontal cones we may have contraction or expansion, but with smaller rates:

$$\lambda_1 \|v\| \leq \|df_x(v)\| \leq \lambda_1^{-1} \|v\|, \qquad \text{for } v \in \mathcal{C}_x^h$$

$$\|df_x(v)\| \leq \mu_1 \|v\|, \qquad \text{for } v \in \mathcal{C}_x^v.$$

The crude analysis of the local dynamics of the semi-indifferent fixed point exhibits the existence of:

- a unique analytic strong stable manifold corresponding to the dissipative eigenvalue μ

$$W^{ss}(0) := \{x \in \mathbb{C}^2 : \lim_{n\to\infty} |\mu|^{-n} \mathsf{dist}(f^n(x), 0) = \mathsf{const.}\},$$

- a (non-unique) center manifold $W^c_{\mathrm{loc}}(0)$ of class C^k for some integer $k \geq 1$, tangent at 0 to the eigenspace E^c of the neutral eigenvalue λ. There exists a ball B (whose size depends on k) centered at 0 in which the center manifold is locally the graph of a C^k function $\varphi_f : E^c \to E^s$ with the properties:

  Local Invariance: $f(W^c_{\mathrm{loc}}) \cap B \subset W^c_{\mathrm{loc}}$.

  Weak Uniqueness: If $f^{-n}(x) \in B\ \forall n \in \mathbb{N}$, then $x \in W^c_{\mathrm{loc}}$.

Hedgehogs in 2D

Theorem (Firsova, Lyubich, Radu, T.)

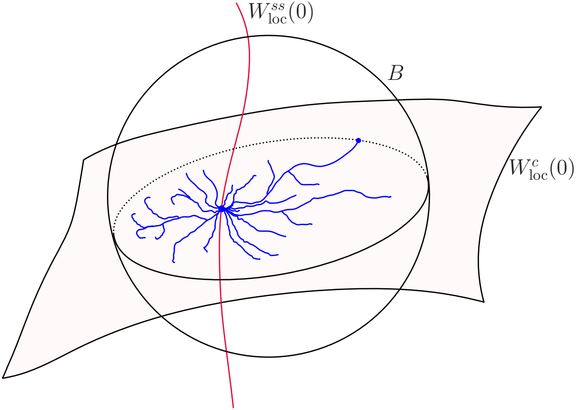
Let f be a germ of holomorphic diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-indifferent fixed point at 0 with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. Consider an open ball $B \subset \mathbb{C}^2$ centered at 0 such that f is partially hyperbolic on a neighborhood B' of $\overline{B}$.
There exists a set $\mathcal{H} \subset \overline{B}$ such that:

a) $\mathcal{H} \Subset W^c_{\text{loc}}(0)$, where $W^c_{\text{loc}}(0)$ is any local center manifold of the fixed point 0 constructed relative to the larger set B'.

b) $\mathcal{H}$ is compact, connected, completely invariant, and full.

c) $0 \in \mathcal{H}$, $\mathcal{H} \cap \partial B \neq \emptyset$.

d) Every point $x \in \mathcal{H}$ has a local strong stable manifold $W^{ss}_{\text{loc}}(x)$, consisting of points from B that converge asymptotically exponentially fast to x, at a rate $\asymp \mu^n$. The strong stable set of $\mathcal{H}$ is laminated by vertical-like holomorphic disks.



Local strong stable & center manifolds.

We can modify the complex structure on the center manifold W^c_{loc} so that the restriction $f|_{W^c_{\mathrm{loc}}}$ becomes analytic.

QC Theorem (Lyubich, Radu, T.)

Let f be a holomorphic germ of diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-neutral fixed point at the origin with eigenvalues λ and μ, where $|\lambda| = 1$ and $|\mu| < 1$. Consider $W^c_{\mathrm{loc}}(0)$ a C^1-smooth local center manifold of the fixed point 0.

There exist neighborhoods W, W' of the origin inside $W^c_{\mathrm{loc}}(0)$ such that $f : W \to W'$ is quasiconformally conjugate to a holomorphic diffeomorphism $h : (\Omega, 0) \to (\Omega', 0)$, $h(z) = \lambda z + \mathcal{O}(z^2)$, where $\Omega, \Omega' \subset \mathbb{C}$.

Moreover, the conjugacy map is holomorphic on the interior of Λ rel $W^c_{\mathrm{loc}}(0)$, where Λ is the set of points that stay in W under all backward iterations of f.

QC Theorem: sketch of proof

- The tangent space $T_x W^c_{\mathrm{loc}}$ at any point $x \in \Lambda$ is a complex line E^c_x of $T_x\mathbb{C}^2$. The line field over Λ is df-invariant, in the sense that $df_x(E^c_x) = E^c_{f(x)}$ for every point $x \in \Lambda$ with $f(x) \in \Lambda$.
- The standard Hermitian structure on $\mathbb{C}^2$ defines a Riemannian metric on the underlying smooth manifold $\mathbb{R}^4$, which restricts to a Riemannian metric on the center manifold W^c_{loc}.
- Every Riemannian metric on W^c_{loc} induces an almost complex structure $J'_x : T_x W^c_{\mathrm{loc}} \to T_x W^c_{\mathrm{loc}}$.
- Let J be the standard almost complex structure on $\mathbb{C}^2 \simeq \mathbb{R}^4$. Note that $J'_x = J_x$ when $x \in \Lambda$.
- Every almost complex structure on a two-dimensional real manifold is integrable.

There exists a (J', i)-holomorphic parametrization

$$\phi : U \subset \mathbb{C} \to W^c_{\mathrm{loc}}.$$

The map f on $W := B \cap W^c_{\mathrm{loc}}$ induces an orientation-preserving C^1-diffeomorphism $g = \phi \circ f \circ \phi^{-1}$ on U.

$$\begin{array}{ccc} W & \xrightarrow{f} & W' \\ \phi \uparrow & & \uparrow \phi \\ U & \xrightarrow{g} & U' \end{array}$$

$$\begin{aligned} W_n &= \text{the set of points from } W \text{ that stay in } W \text{ under} \\ &\quad\ \text{the first } n \text{ backward iterations of } f. \\ U_n &= \phi^{-1}(W_n) \quad \text{and} \quad U_\infty = \phi^{-1}(\Lambda). \end{aligned}$$

$$\bar{\partial}_{J'} f := \frac{1}{2}(df_\xi + J'_{f(\xi)} \circ df_\xi \circ J'_\xi).$$

Lemma

$\bar{\partial}_{J'} f = 0$ *on* Λ.

Lemma (Estimating the $\bar{\partial}_{J'}$-derivative of f)

There exist $\rho < 1$ *and a constant* C *such that for every* $n \geq 1$,

$$\|\bar{\partial}_{J'} f\|_{W_n} < C\rho^n.$$

We can transport the estimates for f to analogous estimates for g.

Let μ_0 denote the 0 Beltrami coefficient of the standard almost complex structure of the plane. Consider the Beltrami coefficient μ on U, given by

$$\mu = \begin{cases} (g^{-n})^* \mu_0 & \text{on } U_n - U_{n+1}, \text{ for } n \geq 0 \\ \mu_0 & \text{on } U_\infty. \end{cases}$$

Recall that $U_1 = U \cap g(U)$ and $U_{-1} = U \cap g^{-1}(U)$.

Lemma

$\|\mu\|_\infty < 1$ *and* μ *is* g^{-1} *invariant, i.e.* $(g^{-1})^* \mu = \mu$ *on* U_1.

Theorem

The map $g^{-1} : U_1 \to U_{-1}$ is quasiconformally conjugate to an analytic map $h(z) = \lambda z + \mathcal{O}(z^2)$.

By the Measurable Riemann Mapping Theorem, there exists a qc homeomorphism $\psi : U \to \mathbb{C}$ fixing the origin with complex dilatation $\mu_\psi = \mu$.

Let $\Omega = \psi(U_{-1})$ and $\Omega' = \psi(U_1)$.
The map $h : \Omega \to \Omega'$, $h = \psi \circ g \circ \psi^{-1}$ is analytic:

$$\begin{tikzcd}
\big(U_1, (g^{-1})^*\mu\big) \arrow[r, "g^{-1}"] \arrow[d, "\psi"'] & (U_{-1}, \mu) \arrow[d, "\psi"] \\
\Omega' \arrow[r, "h^{-1}"] & \Omega
\end{tikzcd}$$

We can use either the hedgehog or a theorem of Gambaudo-Le Calvez-Pécou to prove that $h'(0) = \lambda$ (the neutral eigenvalue).

Dynamical consequences in 2D

Theorem (Lyubich, Radu, T.)

Let f be a germ of holomorphic diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-Cremer fixed point at 0 with eigenvalues $\lambda = e^{2\pi i\alpha}$ and μ, with $|\mu| < 1$. Suppose $(p_n/q_n)_{n\geq 1}$ are the convergents of the continued fraction of α. Let $\mathcal{H}$ be a hedgehog for f. There exists a subsequence $(n_k)_{k\geq 1}$ such that the iterates $(f^{q_{n_k}})_{k\geq 1}$ converge uniformly on $\mathcal{H}$ to the identity.

Corollary

The dynamics on the hedgehog $\mathcal{H}$ is recurrent. The hedgehog does not contain other periodic points except 0.

Corollary

If $z \notin W^{ss}(0)$ then the orbit of z does not converge to 0. Moreover, $W^{ss}_{\mathrm{loc}}(\mathcal{H}) = W^{s}_{\mathrm{loc}}(\mathcal{H})$.

Theorem (Lyubich, Radu, T.)

Let f be a holomorphic germ of diffeomorphisms of $(\mathbb{C}^2, 0)$ with an isolated semi-neutral fixed point at the origin. Let $\mathcal{H}$ be a hedgehog for f. Then $0 \in \mathrm{int}^c(\mathcal{H})$ if and only if f is analytically conjugate to a linear cocycle $\tilde{f}$ given by

$$\tilde{f}(x, y) = (\lambda x, \mu(x) y)\,,$$

where $\mu(x) = \mu + \mathcal{O}(x)$ is a holomorphic function.

Corollary

Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with a semi-neutral fixed point at the origin. Then $0 \in \mathrm{int}^c(\mathcal{H})$ if and only if f is linearizable.

Theorem (Lyubich, Radu, T.)

Let f be a dissipative polynomial diffeomorphism of $\mathbb{C}^2$ with an irrationally semi-indifferent fixed point at 0. Suppose f is not linearizable in a neighborhood of the origin. Let $\mathcal{H}$ be a hedgehog for f. Then $\mathcal{H} \subset J^*$ (the closure of the saddle periodic points for f) and there are no wandering domains converging to $\mathcal{H}$.

Theorem (Ueda)

Let f be a germ of dissipative holomorphic diffeomorphisms of $(\mathbb{C}^2, 0)$ with a semi-parabolic fixed point at 0 with an eigenvalue $\lambda = e^{2\pi i p/q}$. Suppose the semi-parabolic multiplicity of 0 is ν.
The set

$$\Sigma_B = \{x \in B - \{0\} : f^{-n}(x) \in B \ \forall n \in \mathbb{N}, \ f^{-n}(x) \to 0\}.$$

consists of ν cycles of q repelling petals. Each repelling petal is the image of an injective holomorphic map $\varphi(x) = (x, k(x))$ from a left half plane of $\mathbb{C}$ into $\mathbb{C}^2$, which satisfies $\varphi(x+1) = f^q(\varphi(x))$. The inverse of φ, denoted by $\varphi^o : \Sigma_B \to \mathbb{C}$ is called an outgoing Fatou coordinate; it satisfies the Abel equation $\varphi^o(f^q) = \varphi^o + 1$.

As a direct consequence of the QC Theorem, we obtain the following generalization of Naishul's Theorem to higher dimensions:

Theorem (Lyubich, Radu, T.)

Let f_1 and f_2 be two holomorphic germs of diffeomorphisms of $(\mathbb{C}^2, 0)$, with a fixed point at the origin with eigenvalues λ_j and μ_j, where $\lambda_j = e^{2\pi i\theta_j}$ and $|\mu_j| < 1$, $j = 1, 2$.

If f_1 and f_2 are topologically conjugate by a homeomorphism $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ with $\varphi(0) = 0$, then $\theta_1 = \pm\theta_2$.

Germs of $(\mathbb{C}^n, 0)$ with a neutral direction

QC Theorem (Lyubich, Radu, T.)

Let f be a holomorphic germ of diffeomorphisms of $(\mathbb{C}^n, 0)$. Suppose df_0 has eigenvalues λ_j, $1 \leq j \leq n$, with $|\lambda_k| = 1$ for some k and $|\lambda_j| \neq 1$ when $j \neq k$. Let $W^c_{\mathrm{loc}}(0)$ be a C^1-smooth local center manifold of the fixed point 0.

There exist neighborhoods W, W' of the origin inside $W^c_{\mathrm{loc}}(0)$ such that $f : W \to W'$ is quasiconformally conjugate to a holomorphic diffeomorphism $h : (\Omega, 0) \to (\Omega', 0)$, $h(z) = \lambda_k z + \mathcal{O}(z^2)$, where $\Omega, \Omega' \subset \mathbb{C}$.

Moreover, the conjugacy map is holomorphic on the interior of Z rel $W^c_{\mathrm{loc}}(0)$, where Z is the set of points that stay in W under all forward and backward iterations of f.