Eremenko's conjecture in complex dynamics

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Joint work with Phil Rippon

The Open University

New Developments in Complex Analysis and Function Theory
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Basic definitions

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The **escaping set** is

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}.$$

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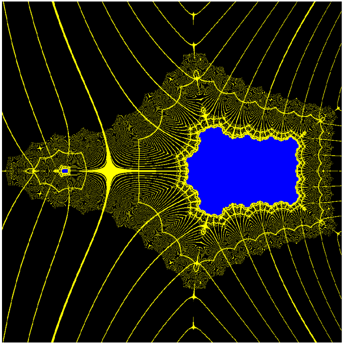
- $F(f)$ is an attracting basin
- $J(f)$ is a Cantor bouquet of curves
- $I(f) \subset J(f)$



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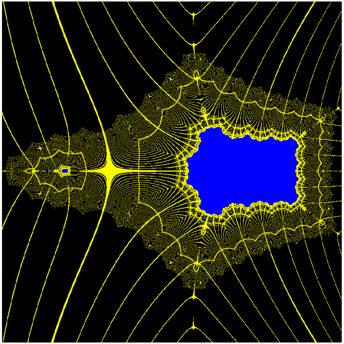
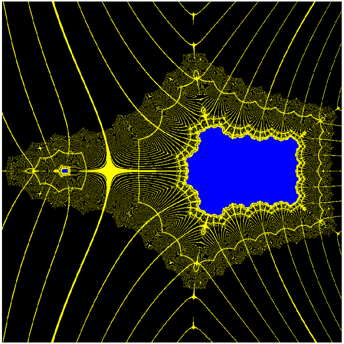
Examples
Spider's web



$f(z) = \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4})$

- $F(f)$ has infinitely many components
- $J(f)$ and $I(f)$ are both spiders' webs

What is a spider's web?



Definition

E is a **spider's web** if

- E is connected;
- there is a sequence of bounded simply connected domains G_n with

$$\partial G_n \subset E, \; G_{n+1} \supset G_n,$$

$$\bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}.$$

Eremenko's conjectures

Theorem (Eremenko, 1989)

If f is transcendental entire then

- $J(f) \cap I(f) \neq \emptyset$;
- $J(f) = \partial I(f)$;
- *all components of* $\overline{I(f)}$ *are unbounded.*

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Conjecture 2 holds for many functions in class $\mathcal{B}$ but fails for others in class $\mathcal{B}$.

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Theorem (cos $\pi\rho$ theorem)

If f has order $\rho < 1/2$ and $\epsilon > 0$, then there exists $c \in (0,1)$ such that, for all large $r > 0$,

$$\log m(t) > (\cos(\pi\rho) - \epsilon)\log M(t), \text{ for some } t \in (r^c, r).$$

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Theorem (Rippon and Stallard, 2005)

All the components of $A_R(f)$ are unbounded and hence $I(f)$ has at least one unbounded component.

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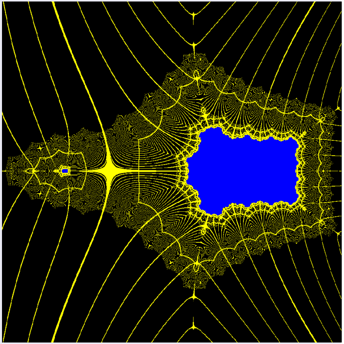

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Hence $A_R(f)$ is a spider's web.

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Theorem (Rippon + Stallard)

Let f be a transcendental entire function. Then there exists $r > R > 0$ such that

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In fact there exists θ such that

$$f(re^{i\theta}) \to 0 \text{ as } r \to \infty.$$

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We deduce that $I(f)$ is a spider's web.

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To hear how this is related to a conjecture of Noel Baker, come to Phil Rippon's talk tomorrow!