2d-shape Analysis using Complex Analysis

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“New Developments
in
Complex Analysis and Function Theory”

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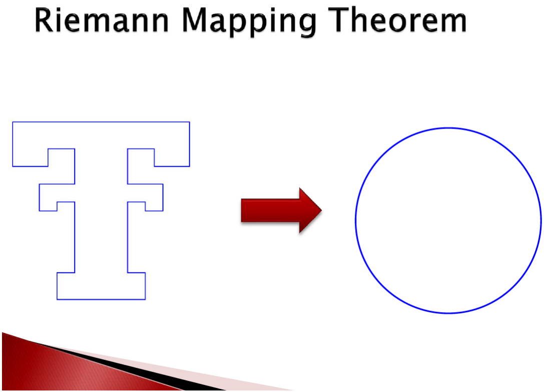
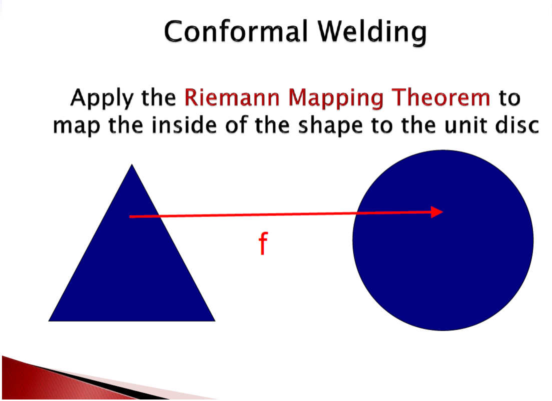
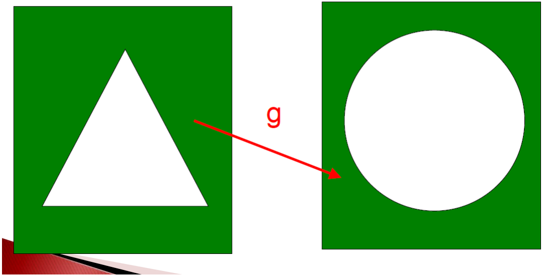




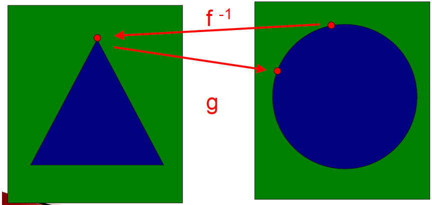
The Riemann mapping function f is continuous on the boundary.

Now map the outside to the outside of the unit disc.

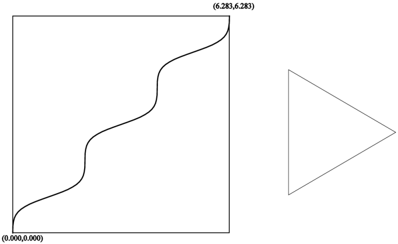


Then function g is also continuous on the boundary.

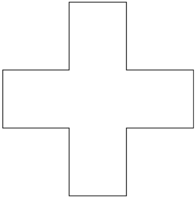
Now $g \circ f^{-1}$ maps the boundary of the unit disc onto itself.

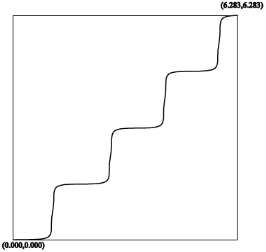


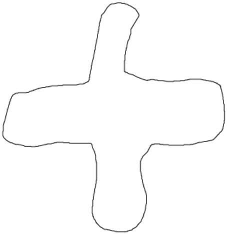
Fingerprint of an Equilateral Triangle

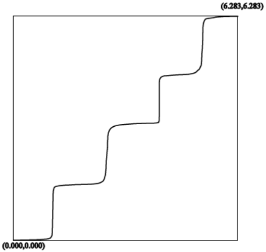


Determining Similarities

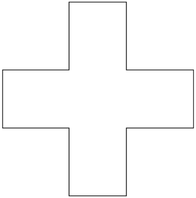


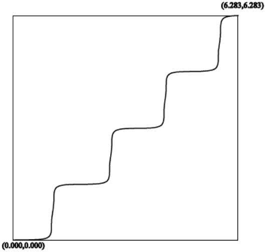


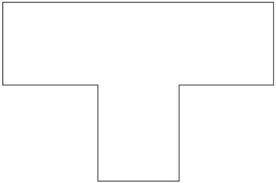


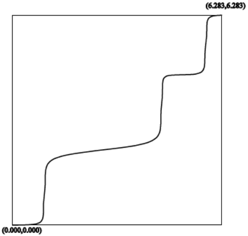


Distinguishing Differences









Let Γ be a Jordan curve in the complex plane $\mathbb{C}$ and let Ω_- and Ω_+ denote the bounded and unbounded components of $\overline{\mathbb{C}} \setminus \Gamma$, where $\overline{\mathbb{C}}$ is the complex sphere. Then Ω_- and Ω_+ are simply connected domains and therefore, by the Riemann mapping theorem, there exist maps $\varphi_- : \mathbb{D} \to \Omega_-$ and $\varphi_+ : \mathbb{D}_+ \to \Omega_+$, where $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk and $\mathbb{D}_+ = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We suppose that φ_+ is normalized by conditions $\varphi_+(\infty) = \infty$, $\varphi'_+(\infty) > 0$, where $\varphi'_+(\infty) = \lim_{z\to\infty} \varphi_+(z)/z$. The latter normalization defines φ_+ uniquely. Each of the maps φ_- and φ_+ extends as a continuous one-to-one function onto the unit circle $\mathbb{T} = \partial\mathbb{D}$.

Therefore, the composition $k = \varphi_+^{-1} \circ \varphi_-$ defines an oriented automorphism of $\mathbb{T}$. Since φ_- is uniquely determined up to a precomposition with a Möbius automorphism of $\mathbb{D}$, the automorphism k is also uniquely determined up to a Möbius automorphism of $\mathbb{D}$, i.e. up to a precomposition with maps

$$\phi(z) = \lambda \frac{z-a}{1-\bar{a}z}, \quad |\lambda| = 1, \quad a \in \mathbb{D}. \tag{1}$$

The equivalence class of the automorphism k under the action of the Möbius group of automorphisms (1) is called the *fingerprint* of Γ. Furthermore, the fingerprint k is invariant under translations and scalings of the curve Γ, i.e. under affine maps $L(z) = az + b$ with $a > 0$, $b \in \mathbb{C}$. The equivalence class of a Jordan curve Γ under the action of affine maps of this form is called the *shape* and Γ is a representative of this shape. Thus, we have a map $\mathcal{F}$ from the set of all shapes into the set of all orientation preserving homeomorphisms of $\mathbb{T}$ onto itself. Let $\mathcal{S}^1$ denote the class of all smooth shapes in $\mathbb{C}$ and let $\mathrm{Diff}(\mathbb{T})$ denote the set of all orientation preserving diffeomorphisms of $\mathbb{T}$.

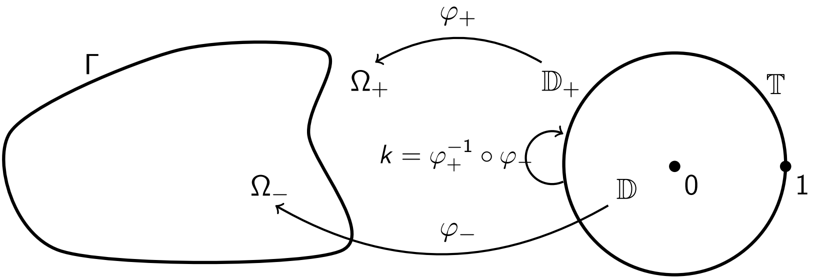


Figure: Jordan curve Γ and complementary domains Ω_- and Ω_+.

The following pioneering result was proved by Alexander A. Kirillov in "Kähler structure on the K -orbits of a group of diffeomorphisms of the circle", *Funktsional. Anal. i Prilozhen.* **21** (1987), no. 2.

Theorem (Kirillov)

The map $\mathcal{F}$ is a bijection between $\mathcal{S}^1$ and Diff($\mathbb{T}$).

In other words, Theorem 1 says that Diff($\mathbb{T}$) parameterizes the set $\mathcal{S}^1$ of all smooth shapes.

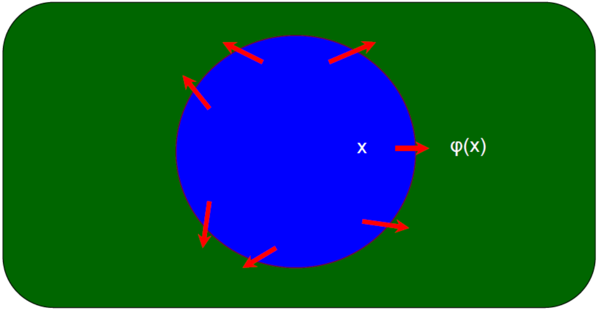
Image Recognition

David Mumford and Eitan Sharon use Fingerprints for 2-D image recognition.

They recognize shapes by their welding maps.

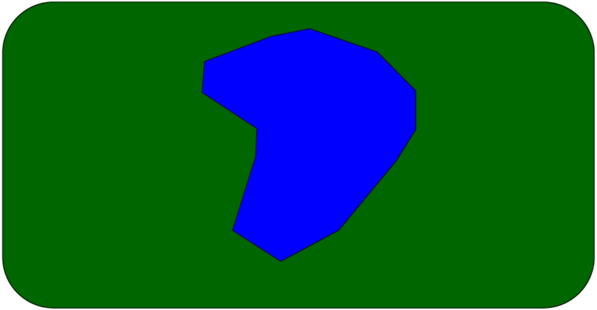
Welding

We can go from fingerprint to shape by gluing the inside and outside of the disc unevenly.



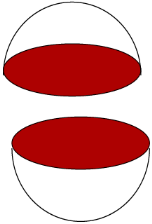
Welding

The welding will distort the circle into the new shape.



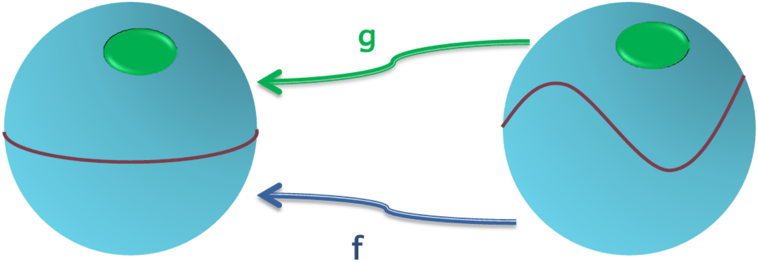
Stereographic Projection

If we project to the sphere, we can see this as gluing two half- spheres together.

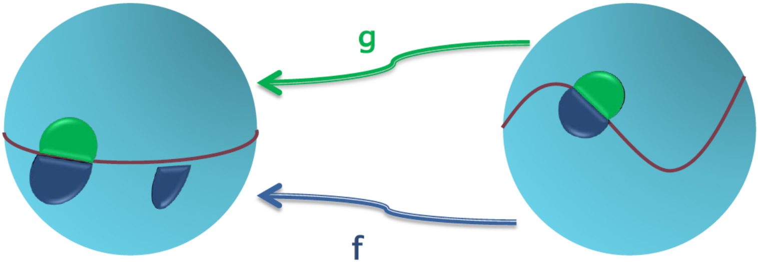


Weld the halves back together by attaching x in the lower half to $\phi(x)$ in the upper half.

Preserving Conformal Structure



Preserving Conformal Structure



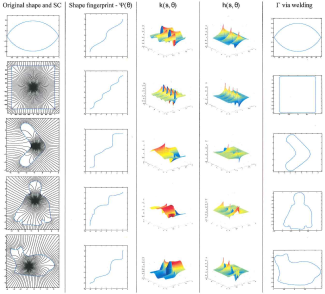
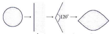


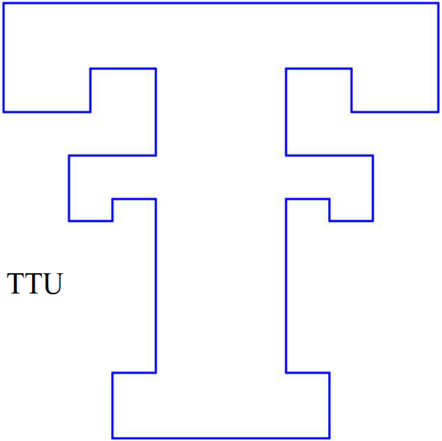
Figure 4: Experimental results. For each shape we present a row of five columns. (From left) 1st column: The conformal mappings Φ_- and Φ_+ carrying the two copies of the unit disc, Δ_- and Δ_+ onto the interior and the exterior of the Γ shape, as explained in Sec. 2.1. The figure illustrates how a homogenous radial grid on Δ_- and Δ_+, made of concentric lines through the origin, is mapped differently into the interior and exterior of the Γ shape. Note the differences in the densities of the radial grid lines, along Γ, between the interior and the exterior maps. This difference in densities along Γ is exactly what is encoded by the diffeomorphisms $\Psi \in G/H$ that match Γ, and is the fingerprint of the shape. 2nd column: Shape fingerprint $\Psi(\theta) : [0, 2\pi] \to [0, 2\pi]$, computed analytically for the first row, and numerically via Schwarz-Christoffel for the last four rows. 3rd column: $k(s, \theta)$, computed over the grid (s_i, θ_j). 4th column: $h(s, \theta)$, conjugated from $k(s, \theta)$ over the grid (s_i, θ_j). 5th column: the shape Γ, as it results from Ψ via welding, up to scale and translation, obtained by drawing $f\,(s = 0)$ in the complex plane.



$z_1 = e^{s+i\theta}$ $\quad z_2 = \frac{1-z_1}{1+z_1}$ $\quad z_3 = z_2^{2/3}$ $\quad z_4 = \frac{1-z_3}{1+z_3}$

Figure 5: Example: The construction of Φ_- - the conformal mapping of the interior of the unit disc onto the interior of the "eye" shape, presented in steps.

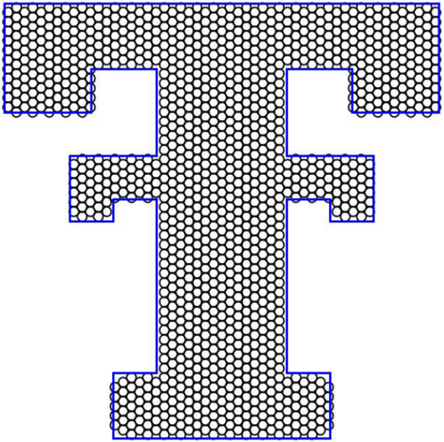
Discrete Conformal Maps



Brock Williams, TTU

Discrete Conformal Maps

Circle Packing



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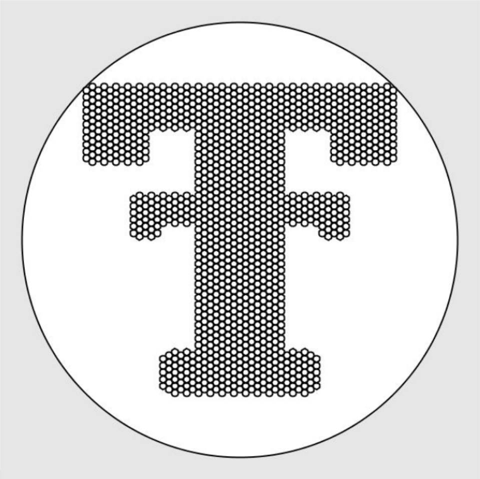

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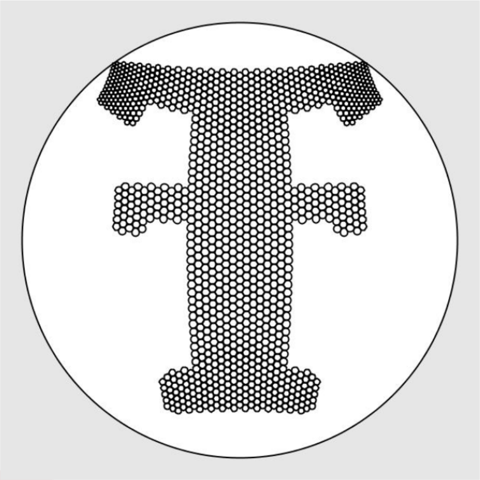

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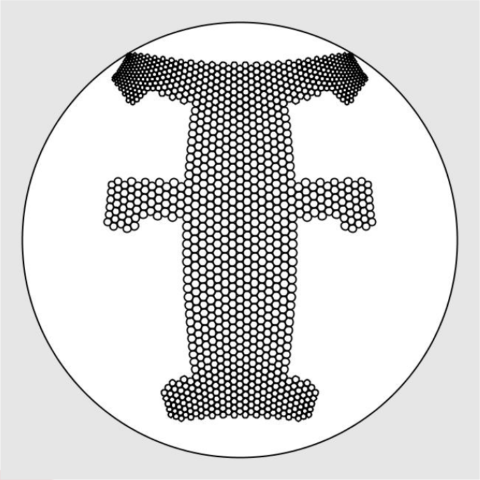

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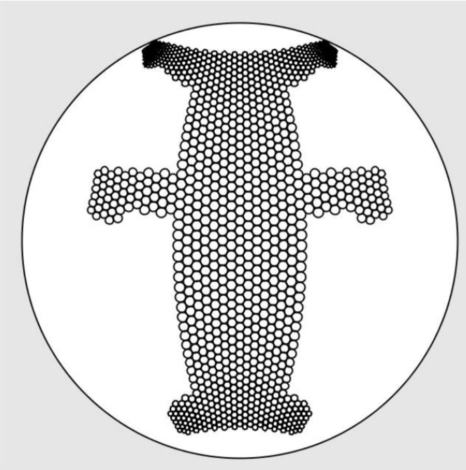

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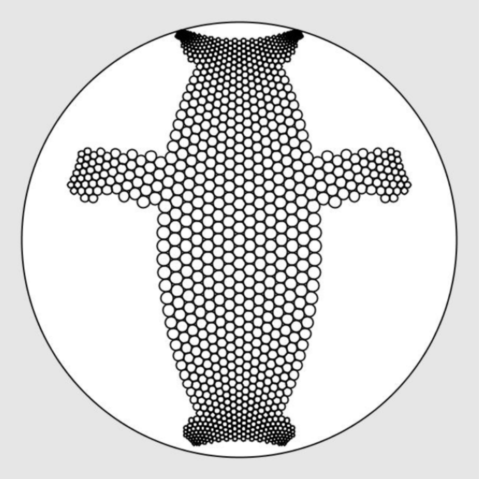

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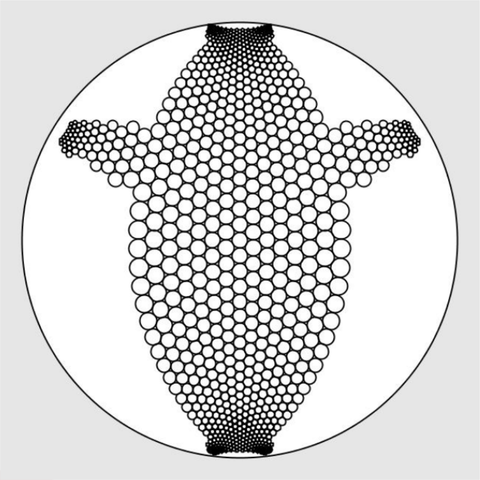

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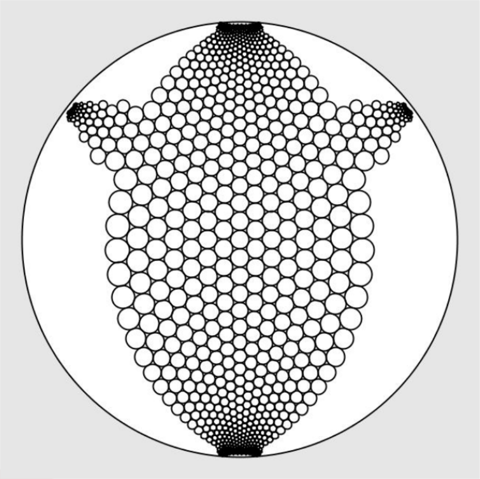

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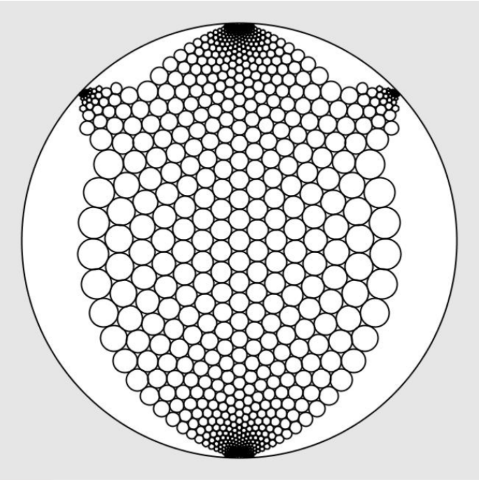

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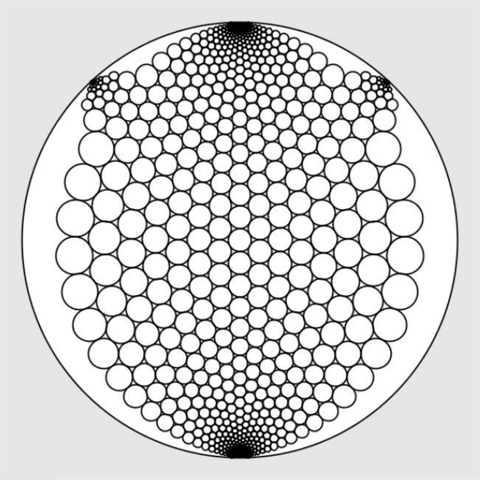

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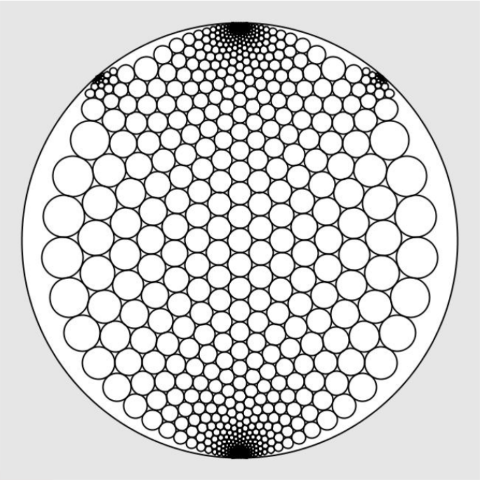

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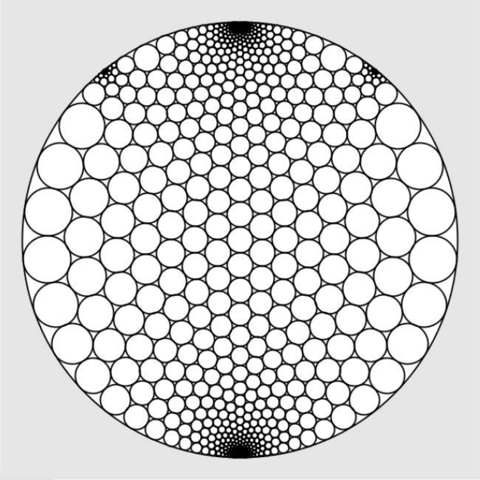

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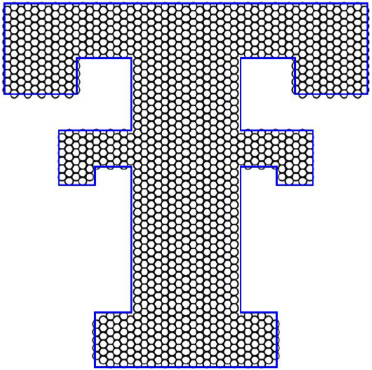
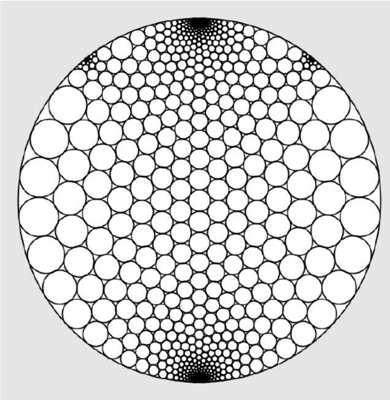
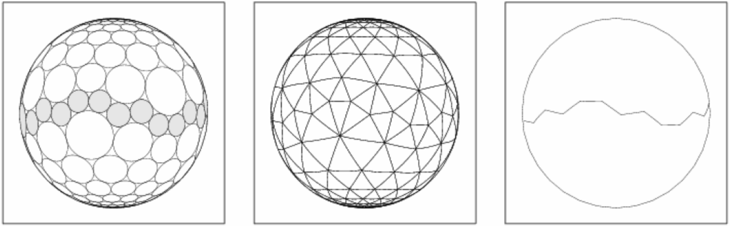




About 900 Circles in Each Packing

Example



Polynomial Lemniscates

The set $\mathrm{Diff}(\mathbb{T})$ is rather big. So, Ebenfeld, Khavinson, and Shapiro studied possible parameterizations of fingerprints of polynomial lemniscates. A lemniscate of a polynomial $P(z)$ at level $c > 0$ is defined as $L_P(c) = \{z : |P(z)| = c\}$. Later on, we will consider also rational lemniscates and lemniscates associated with nonconstant meromorphic functions. The class of polynomial lemniscates is much smaller that $\mathcal{S}^1$, nevertheless, by the Hilbert theorem it still can be used to approximate any Jordan shape. The following theorem of Ebenfeld, Khavinson, and Shapiro characterizes exactly which elements of $\mathrm{Diff}(\mathbb{T})$ appear to be the fingerprints of polynomial lemniscates.

Theorem (P. Ebenfelt, D. Khavinson, Harold Shapiro)

Let $P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0$ be a polynomial of degree n with $c_n > 0$ such that $L_P(1)$ is analytic and connected and let $k : \mathbb{T} \to \mathbb{T}$ be a fingerprint of $L_P(1)$. Then $k(z)$ is given by the equation

$$(k(z))^n = B(z), \tag{2}$$

where $B(z)$ is a Blaschke product of degree n,

$$B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z},$$

with some real α, where $a_k = \varphi_-^{-1}(\zeta_k)$ and $\zeta_1, \ldots, \zeta_n$ are the zeroes of $P(z)$ counting multiplicities.

Conversely, given any Blaschke product of degree n, there is a polynomial $P(z)$ of the same degree whose lemniscate $L_P(1)$ is analytic and connected and has $k(z) = B(z)^{1/n}$ as its fingerprint. Moreover, $P(z)$ is unique up to precomposition with an affine map of the form $L(z) = az + b$ with $a > 0$ and $b \in \mathbb{C}$.

Peter Ebenfelt, Dima Khavinson and Harold Shapiro suggested that their method can be extended further to study lemniscates of rational functions.

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Their proof of previous theorem is rather involved. A shorter proof was given by Malik Younsi who also proved a counterpart of Ebenfelt-Khavinson-Shapiro for the case of rational lemniscates.

Fingerprints of Rational Lemniscates

Theorem (M. Younsi)

Let $R(z)$ be a rational function of degree n with $R(\infty) = \infty$ such that its lemniscate $L_R(1) = \{z : |R(z)| = 1\}$ is analytic and connected and let $k : \mathbb{T} \to \mathbb{T}$ be a fingerprint of $L_R(1)$. Then $k(z)$ is given by a solution to the functional equation

$$A \circ k = B, \tag{3}$$

where $A(z)$ and $B(z)$ are Blaschke products of degree n and $A(\infty) = \infty$.
Conversely, given any solution $k(z)$ to a functional equation $A \circ k = B$, where $A(z)$ and $B(z)$ are Blaschke products of degree n and $A(\infty) = \infty$, there exist a rational function $R(z)$ of degree n with $R(\infty) = \infty$ whose lemniscate $L_R(1)$ is analytic and connected and has $k(z)$ as its fingerprint.

Basics of Quadratic Differentials

A quadratic differential on a domain $D \subset \overline{\mathbb{C}}$ is a differential form $Q(z)\,dz^2$ with meromorphic $Q(z)$ and with conformal transformation rule

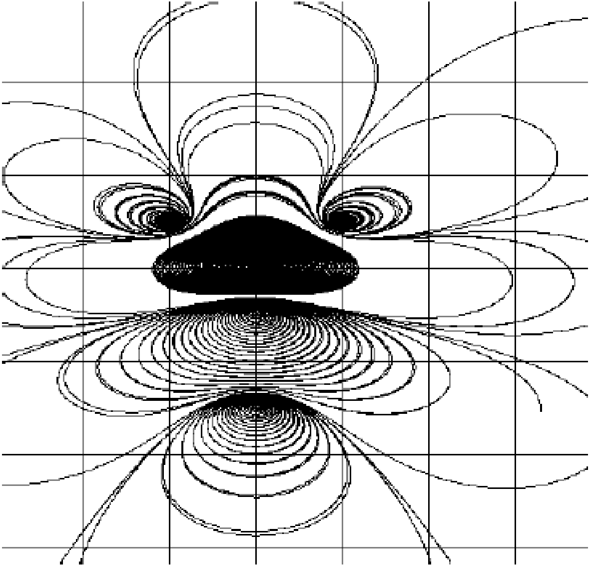
$$Q_1(\zeta)\,d\zeta^2 = Q(\varphi(z))\,(\varphi'(z))^2\,dz^2,$$

where $\zeta = \varphi(z)$ is a conformal map from D onto a domain G in the extended plane of the parameter ζ. Then zeros and poles of $Q(z)$ are critical points of $Q(z)\,dz^2$, in particular, zeros and simple poles are finite critical points and poles of order greater than 1 are infinite critical points of $Q(z)\,dz^2$. A trajectory (respectively, orthogonal trajectory) of $Q(z)\,dz^2$ is a closed analytic Jordan curve or maximal open analytic arc $\gamma \subset D$ such that

$$Q(z)\,dz^2 > 0 \quad \text{along } \gamma \qquad (\text{respectively, } Q(z)\,dz^2 < 0 \quad \text{along } \gamma).$$

Trajectories

A trajectory γ is called *critical* if at least one of its end points is a finite critical point of $Q(z)\,dz^2$. If γ is a rectifiable arc in D then its Q-length is defined by $|\gamma|_Q = \int_\gamma |Q(z)|^{1/2}\,|dz|$. The important property of quadratic differentials is that transformation rule respects trajectories, orthogonal trajectories and their Q-lengthes, as well as it respects critical points together with there multiplicities and trajectory structure nearby.



Fingerprints and Quadratic Differentials

Returning to fingerprints, suppose that Γ is a piece-wise smooth Jordan curve in the plane $\mathbb{C}$ of the parameter ζ and that $Q(\zeta)\,d\zeta^2$ is a quadratic differential on some neighborhood G of Γ. The best case scenario is when $G = \overline{\mathbb{C}}$. More generally, we may assume, without loss of generality, that G is a doubly connected domain bounded by Jordan analytic curves and that Γ separates boundary components of G. Let $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ denote pullbacks of $Q(\zeta)\,d\zeta^2$ under the conformal maps $\zeta = \varphi_-(z)$ and $\zeta = \varphi_+(z)$ defined in Section 1 and let $\tau_-(\zeta) = \varphi_-^{-1}(\zeta)$ and $\tau_+(\zeta) = \varphi_+^{-1}(\zeta)$. Then

$$Q(\zeta)\,d\zeta^2 = Q_-(\tau_-(\zeta))(\tau_-'(\zeta))^2\,d\zeta^2 \quad \text{for } \zeta \in \Omega_- \cap \Gamma$$

and

$$Q(\zeta)\,d\zeta^2 = Q_+(\tau_+(\zeta))(\tau_+'(\zeta))^2\,d\zeta^2 \quad \text{for } \zeta \in \Omega_+ \cap \Gamma.$$

Since Γ is piece-wise smooth it follows that each of the maps $\tau_-(\zeta)$ and $\tau_+(\zeta)$ and their derivatives $\tau'_-(\zeta)$ and $\tau'_+(\zeta)$ can be extended by continuity to any smooth arc of Γ. If Γ is smooth at ζ then previous equations imply that

$$Q_-(\tau_-(\zeta))(\tau'_-(\zeta))^2\, d\zeta^2 = Q_+(\tau_+(\zeta))(\tau'_+(\zeta))^2\, d\zeta^2.$$

Changing variable via $\zeta = \varphi_-(z)$, we obtain equivalent equation

$$Q_-(z)\, dz^2 = Q_+(k(z))(k'(z))^2\, dz^2,$$

which holds for all points $z \in \mathbb{T}$ such that $\varphi_-(z)$ belongs to a smooth arc of Γ. Here $k = \varphi_+^{-1} \circ \varphi_-$ is homeomorphism from $\mathbb{T}$ onto itself and therefore it is a fingerprint of Γ.

Taking square roots of both sides and then integrating along the unit circle, we obtain

$$\int_{z_0}^{z} \sqrt{Q_+(k(z))}k'(z)\,dz = \int_{z_0}^{z} \sqrt{Q_-(z)}\,dz,$$

where

$$z_0 = e^{i\theta_0}, \quad z = e^{i\theta} \quad \text{with } 0 \le \theta_0 < 2\pi, \quad \theta_0 \le \theta \le \theta_0 + 2\pi$$

and integration is taken along a circular arc $l(\theta_0, \theta) = \{e^{it} : \theta_0 \le t \le \theta\}$.

Main Lemma

The following lemma summarizes simple observations made above for the case when infinite critical points are absent.

Lemma

Let Γ be a piece-wise smooth Jordan curve and let $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ be pullbacks of the quadratic differential $Q(\zeta)\,d\zeta^2$ introduced above. If Γ does not contain infinite critical points of $Q(\zeta)\,d\zeta^2$ then the fingerprint $k : \mathbb{T} \to \mathbb{T}$ of Γ is given by a solution to the functional equation

$$\mathcal{A} \circ k = \mathcal{B}, \qquad (*)$$

where

$$\mathcal{A}(z) = \int_{k(z_0)}^{z} \sqrt{Q_+(z)}\,dz, \quad \mathcal{B}(z) = \int_{z_0}^{z} \sqrt{Q_-(z)}\,dz \qquad (**)$$

appropriately chosen branches of the radicals.

Some Difficulties

We admit here that our main equation is of little practical use unless we know how to find functions $\mathcal{A}(z)$ and $\mathcal{B}(z)$ or, equivalently, how to construct quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$. In general, such construction looks problematic. Rare cases when this is possible include cases of polynomial and rational lemniscates. Below, we explain how to find a general form of the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ assuming some additional assumptions.

Namely, we suppose now that a quadratic differential $Q(\zeta)\,d\zeta^2$ is defined on the whole complex sphere $\overline{\mathbb{C}}$. Then, of course, $Q(\zeta)$ is a rational function. Furthermore, we suppose that Γ is a Jordan curve consisting of a finite number of arcs, $\gamma_1, \ldots, \gamma_m$, of trajectories and/or orthogonal trajectories of $Q(\zeta)\,d\zeta$ and their end points.

Possible structure of Γ near a point $\zeta_0 \in \Gamma$ where two arcs meet.

(1) If ζ_0 is a regular point of $Q(\zeta)\,d\zeta^2$, then one of the arcs γ_1 and γ_2 is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. In this case, γ_1 and γ_2 form a corner of opening $\pi/2$ with respect to one of the domains Ω_- and Ω_+ and a corner of opening $3\pi/2$ with respect to the other one.

(2) If ζ_0 is a zero of $Q(\zeta)\,d\zeta^2$ of order n then arcs γ_1 and γ_2 form an angle of opening $\pi k/(n+2)$ with some integer k, $0 < k < 2(n+2)$. Moreover, if k is odd then one of the arcs γ_1, γ_2 is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. If k is even then both γ_1 and γ_2 are either arcs of trajectories or arcs of orthogonal trajectories.

(3) If ζ_0 is a simple pole, then one of the arcs γ_1 and γ_2 is an arc of a trajectory and the other one is an arc of an orthogonal trajectory and Γ is analytic at ζ_0.

(4) If ζ_0 is a pole of order two then both γ_1 and γ_2 are either arcs of trajectories or arcs of orthogonal trajectories. If they are arcs of trajectories then $Q(\zeta)\,dz^2$ has radial or spiral structure of trajectories near ζ_0 and if they are arcs of orthogonal trajectories $Q(\zeta)\,dz^2$ has circular or spiral structure of trajectories. In case of radial or circular structure of trajectories, γ_1 and γ_2 can form any angle α, $0 < \alpha < 2\pi$. In case of spiral structure of trajectories, γ_1 and γ_2 in a neighborhood of ζ_0 look like logarithmic spirals and do not form any angle at ζ_0.

(5) Finally, if ζ_0 is a pole of order $n \geq 3$, then γ_1 and γ_2 can form any angle of opening $\pi k/(n-2)$ with some integer k, $0 \leq k \leq 2(n-2)$. Moreover, if k is odd then one of the arcs γ_1, γ_2 is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. If k is even then both γ_1 and γ_2 are either arcs of trajectories or arcs of orthogonal trajectories.

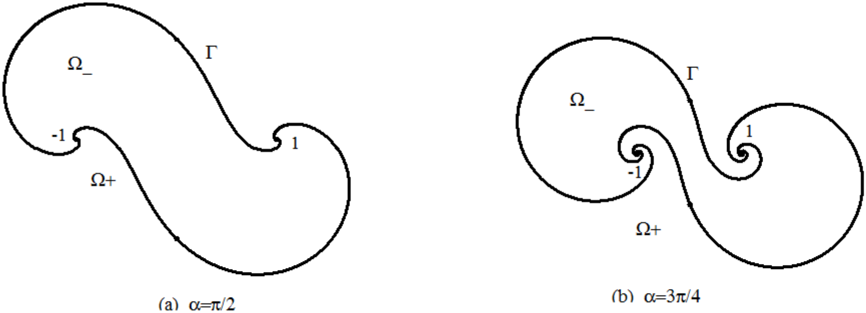


Figure: Γ consisting of two spirals with different α.

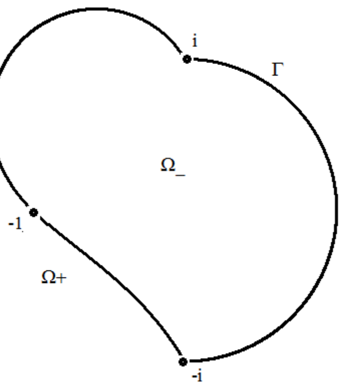


Figure: Γ consisting of three critical trajectories.

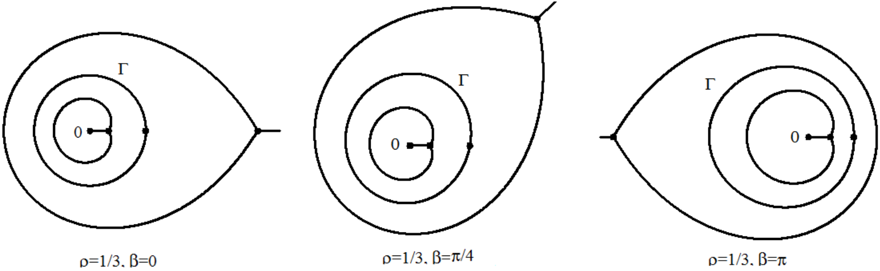


Figure: Γ consisting of one regular trajectory.

Possible critical points of $Q_-(z)\,dz^2$ on $\mathbb{T}$.

(a) If ζ_0 is a regular point of $Q(\zeta)\,d\zeta^2$ where Γ_- forms an angle $\pi/2$, then $z_0 = \tau_-(\zeta_0)$ is a simple pole of $Q_-(z)\,dz^2$.

(b) If ζ_0 is a regular point of $Q(\zeta)\,d\zeta^2$ where Γ_- forms an angle $3\pi/2$, then $z_0 = \tau_-(\zeta_0)$ is a simple zero of $Q_-(z)\,dz^2$.

(c) If ζ_0 is a zero of $Q(\zeta)\,d\zeta^2$ of order n where Γ_- forms an angle $\frac{\pi k}{n+2}$, then $z_0 = \tau_-(\zeta_0)$ is a simple pole of $Q_-(z)\,dz^2$ if $k = 1$, a regular point if $k = 2$, and a zero of order $k - 2$ if $3 \le k \le 2n + 3$.

(d) If ζ_0 is a simple pole of $Q(\zeta)\,dz^2$, then $z_0 = \varphi_-(\zeta_0)$ is a simple pole of $Q_-(z)\,dz^2$.

(e) If ζ_0 is a pole of order 2 of $Q(\zeta)\,dz^2$ and γ_1, γ_2 are arcs of orthogonal trajectories, then $z_0 = \varphi_-(\zeta_0)$ is a pole of $Q_-(z)\,dz^2$ of order 2 with the circular trajectory structure.

(f) If ζ_0 is a pole of order 2 of $Q(\zeta)\,dz^2$ and γ_1, γ_2 are arcs of trajectories, then $z_0 = \varphi_-(\zeta_0)$ is a pole of $Q_-(z)\,dz^2$ of order 2 with a radial trajectory structure.

(g) If ζ_0 is a pole of order $n \geq 3$ of $Q(\zeta)\,dz^2$ where Γ_- forms an angle $\frac{\pi k}{n-2}$, then $z_0 = \varphi_-(\zeta_0)$ is a pole of $Q_-(z)\,dz^2$ of order $k+2$, $0 \leq k \leq 2(n-2)$. In particular, if $k = 0$ and γ_1, γ_2 are arcs of trajectories, then z_0 is a pole of $Q_-(z)\,dz^2$ of order 2 with radial trajectory structure and if $k = 0$ and γ_1, γ_2 are arcs of orthogonal trajectories, then z_0 is a pole of $Q_-(z)\,dz^2$ of order 2 with circular trajectory structure.

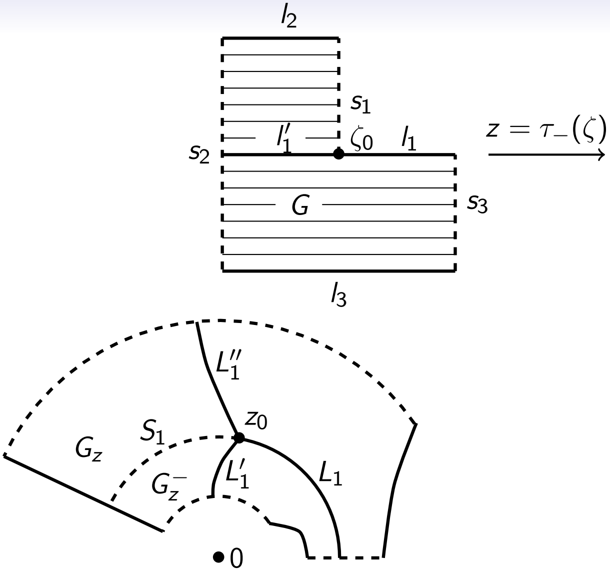


Figure: Trajectory structure in the case (b).

Unit circle consisting of arcs of trajectories

Now, let $B^0_-(z) = \prod_{k=1}^{n_0^-}(z - c_k)(1 - \bar{c}_k z)$ and $B^\infty_-(z) = \prod_{k=1}^{n_\infty^-}(z - p_k)(1 - \bar{p}_k z)$, where the products are taken over all zeros (counting multiplicity) of $Q_-(z)\,dz^2$ in the unit disk $\mathbb{D}$ and over all poles (counting multiplicity) of $Q_-(z)\,dz^2$ in the unit disk $\mathbb{D}$, respectively. Also, let $P^0_-(z) = \prod_{k=1}^{m_0^-} e^{-i(\pi+\alpha_k)/2}(z - e^{i\alpha_k})$ and $P^\infty_-(z) = \prod_{k=1}^{m_\infty^-} e^{-i(\pi+\beta_k)/2}(z - e^{i\beta_k})$, where the products are taken over all zeros (counting multiplicity) of $Q_-(z)\,dz^2$ on the unit circle $\mathbb{T}$ and over all poles (counting multiplicity) of $Q_-(z)\,dz^2$ on the unit circle $\mathbb{T}$, respectively. Let $B^0_+(z)$, $B^\infty_+(z)$, $P^0_+(z)$, and $P^\infty_+(z)$ denote similar products for the quadratic differential $Q_+(z)\,dz^2$ and let n_0^+, n_∞^+, m_0^+, m_∞^+ denote the number of terms in the corresponding products.

Forms of Quadratic Differentials

In notations introduced above, the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ can be written as

$$Q_-(z)\,dz^2 = C_- \frac{P_-^0(z)B_-^0(z)}{P_-^\infty(z)B_-^\infty(z)}\,dz^2, \quad Q_+(z)\,dz^2 = C_+ \frac{P_+^0(z)B_+^0(z)}{P_+^\infty(z)B_+^\infty(z)}\,dz^2$$

with some real nonzero constants C_- and C_+.

Main Theorem on Fingerprints

We note that our conditions imply that quadratic differentials defined above are real on the unit circle except corresponding sets of their critical points (which may be empty). Combining this with our main lemma, we obtain the following.

Theorem

Let $Q(\zeta)\,d\zeta^2$ be a quadratic differential on $\overline{\mathbb{C}}$ and let Γ be a Jordan curve free of infinite critical points of $Q(\zeta)\,d\zeta^2$ which consists of a finite number of arcs of trajectories and/or orthogonal trajectories of $Q(\zeta)\,d\zeta$ and their end points. Then the fingerprint $k : \mathbb{T} \to \mathbb{T}$ of Γ is given by a solution to the functional equation (*) with $\mathcal{A}(z)$ and $\mathcal{B}(z)$ defined by (**) and $Q_-(z)\,z^2$ and $Q_+(z)\,dz^2$ defined above.

Coordinated Quadratic Differentials

The converse statement for this theorem, similar to converse statements of theorems proved by P. Ebenfelt, D. Khavinson and H.S. Shapiro or M. Younsi will require additional restrictions which are discussed below.

Definition

We will say that quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ are coordinated on $\mathbb{T}$ if their systems of arcs $\alpha_1^-,\ldots,\alpha_{k_-}^-$ and $\alpha_1^+,\ldots,\alpha_{k_+}^+$ have the same number of arcs and if they can be enumerated in the counter clock-wise direction on $\mathbb{T}$ in such a way that the following conditions are satisfied:

(a) $Q_-(z)\,dz^2 \geq 0$ on α_j^- if and only if $Q_+(z)\,dz^2 \geq 0$ on α_j^+.

(b) z_j^- is an infinite critical point of $Q_-(z)\,dz^2$ if and only if z_j^+ is an infinite critical point of $Q_+(z)\,dz^2$.

(c) If $|\alpha_j^-|_{Q_-}$ is finite then $|\alpha_j^-|_{Q_-} = |\alpha_j^+|_{Q_+}$.

Converse Statement

Theorem

Suppose that the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ are coordinated on $\mathbb{T}$. Then there is a quadratic differential $Q(\zeta)\,d\zeta^2$ defined on $\overline{\mathbb{C}}$ and a closed Jordan curve Γ consisting of arcs of trajectories and/or orthogonal trajectories of $Q(\zeta)\,d\zeta$ such that the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ are pullbacks of the quadratic differential $Q(\zeta)\,d\zeta^2$ under appropriate mappings φ_- and φ_+ associated with Γ.

Lemniscates.

Let $f(z)$ be a nonconstant meromorphic function on a domain $D \subset \overline{\mathbb{C}}$. For $0 \leq c \leq \infty$, the lemniscate of $f(z)$ at level c is defined by equation

$$L_f(c) = \{z \in D : |f(z)| = c\}.$$

Lemniscates and Tangent Vectors.

Let $L'_f(c)$ be a lemniscate component having a tangent vector dz at its point z. The gradient of the real valued function $\log|f(z)|$ at z can be calculated as follows:

$$\mathrm{grad}(\log|f(z)|) = 2\frac{\partial}{\partial \overline{z}} \log|f(z)| = \overline{\left(\frac{f'(z)}{f(z)}\right)}.$$

Hence, the tangent vector dz to $L'_f(c)$ at z is given by

$$dz = i\overline{\left(\frac{f'(z)}{f(z)}\right)}.$$

Lemniscates and quadratic differentials.

Multiplying both sides of previous equation by $\frac{i}{2\pi}\frac{f'(z)}{f(z)}$ and then squaring, we obtain

$$-\frac{1}{4\pi^2}\left(\frac{f'(z)}{f(z)}\right)^2 dz^2 = \frac{1}{4\pi^2}\left|\frac{f'(z)}{f(z)}\right|^2.$$

The left hand-side defines a quadratic differential, which will be denoted by $Q_f(z)\,dz^2$; i.e.,

$$Q_f(z)\,dz^2 = -\frac{1}{4\pi^2}\left(\frac{f'(z)}{f(z)}\right)^2 dz^2$$

Now this equation shows that

$$Q_f(z)\,dz^2 > 0,$$

if dz is a tangent vector to the lemniscate of $f(z)$ passing through z.

Special Structure of Trajectories.

Since the quadratic differential $Q_f(z)\,dz^2$ is generated by the logarithmic derivative its trajectory structure is not of a general form.
In particular, that trajectory structure of $Q_f(z)\,dz^2$ does not include end domains and strip domains having poles on their boundaries in D. Also, the trajectory structure of $Q_f(z)\,dz^2$ does not include density domains since otherwise some level set of $f(z)$ would be dense in such a domain. Then, $|f(z)|$ must be constant on a density domain and therefore $f(z)$ must be constant on D contradicting our assumption.

(a) Cartesian polygonal curves. By a Cartesian polygonal curve we understand a Jordan curve consisting of a finite number of horizontal and vertical segments. Any such curve Γ is a boundary of a *standard polygon* Ω_- having an even number of sides and even number of vertices, $v_1, \dots, v_{2n}$. We suppose here that vertices are always oriented in the counterclockwise direction and that $v_{2n+1} = v_1$, $v_0 = v_{2n}$.
The horizontal and vertical sides of Ω_- are arcs of trajectories and, respectively, arcs of orthogonal trajectories of the quadratic differential $Q(\zeta)\,d\zeta^2 = 1 \cdot d\zeta^2$. Transplanting this quadratic differential via the mapping $\varphi_- : \mathbb{D} \to \Omega_-$, we obtain the following quadratic differential:

$$Q_-(z)\,dz^2 = C_- e^{i\gamma_-} \prod_{k=1}^{2n} (z - e^{i\beta_k^-})^{2(\alpha_k - 1)}\,dz^2, \qquad z \in \mathbb{D}, \quad (4)$$

with some $C_- > 0$, $\gamma_- \in \mathbb{R}$, and with $e^{i\beta_k^-} = \tau_-(v_k)$, where $0 \le \beta_1^- < \beta_2^- < \cdots < \beta_{2n}^- < \beta_1^- + 2\pi$.

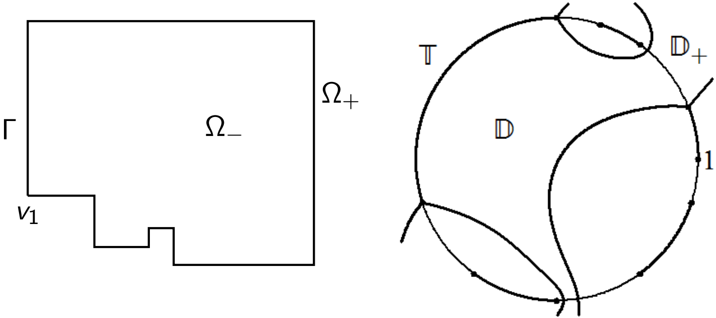


Figure: Cartesian polygonal curve and critical trajectories of $Q_-(z)\,dz^2$.

(b) Polar polygonal curves. We start with the quadratic differential

$$Q(\zeta)\,d\zeta^2 = -\frac{d\zeta^2}{\zeta^2}. \tag{5}$$

Then the radial segments of the form $\{\zeta = re^{i\alpha} : \ r_1 \le r \le r_2\}$ with some $\alpha \in \mathbb{R}$ and $0 < r_1 < r_2 < \infty$ are closed arcs on the orthogonal trajectories of $Q(\zeta)\,d\zeta^2$ and the closed arcs of circles centered at $\zeta = 0$ are closed arcs on the trajectories of $Q(\zeta)\,d\zeta^2$. By a polar polygonal curve Γ we mean a closed Jordan curve bounded by a finite number of radial segments and circular arcs as above.

Transplanting $Q(\zeta)\,d\zeta^2$ via the mapping $\varphi_- : \mathbb{D} \to \Omega_-$ and assuming that $\varphi(0) = 0$, we obtain the following quadratic differential:

$$Q_-(z)\,dz^2 = -C_- e^{i\gamma_-} z^{-2} \prod_{k=1}^{2n} (z - e^{i\beta_k^-})^{2(\alpha_k - 1)}\,dz^2, \qquad z \in \mathbb{D}, \tag{6}$$

where $e^{i\beta_k^-} = \tau_-(v_k)$ with $0 \le \beta_1^- < \beta_2^- < \cdots < \beta_{2n}^- < \beta_1^- + 2\pi$.

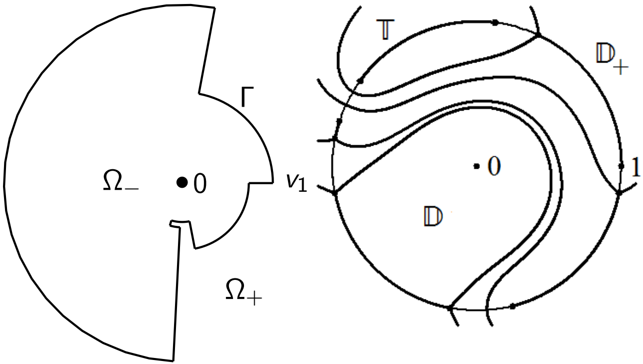


Figure: Polar polygonal curve and critical trajectories of $Q_-(z)\,dz^2$.

Equation

$$\frac{\int_{\beta_{k-1}^+}^{\beta_k^+} \prod_{j=1}^{2n} \left(e^{i\theta} - e^{i\beta_j^+}\right)^{1-\alpha_j} \, e^{-i\theta} \, d\theta}{\int_{\beta_{k-1}^-}^{\beta_k^-} \prod_{j=1}^{2n} \left(e^{i\theta} - e^{i\beta_j^-}\right)^{\alpha_j - 1} \, e^{i\theta} \, d\theta} = Ce^{i\gamma},$$

gives necessary and sufficient conditions which guarantee that the Schwarz-Christoffel integrals representing functions φ_- and φ_+ define one-to-one mappings from $\mathbb{D}$ and $\mathbb{D}_+$ onto polygons Ω_- and Ω_+, respectively. Experts know that a similar fact holds true for the Schwarz-Christoffel mappings from $\mathbb{D}$ and $\mathbb{D}_+$ onto any two complementary polygons with common Jordan boundary. Surprisingly, this author was not able to find the latter fact in standard textbooks on Complex Analysis. Thus, we state it here.

Theorem

For $n \geq 3$, let $0 \leq \beta_1^- < \beta_1^- < \cdots < \beta_n^- < \beta_1^- + 2\pi$ and let $0 < \alpha_k < 2$, $k = 1, 2, \ldots, n$, be such that $\sum_{k=1}^n \alpha_k = n - 2$. Then the Schwarz-Christoffel integral

$$F(z) = \int_0^z \prod_{k=1}^n (\tau - e^{i\beta_k^-})^{\alpha_k - 1} \, d\tau$$

maps $\mathbb{D}$ conformally and one-to-one onto some polygon if and only if there are points $z_k^+ = e^{i\beta_k^+}$ with $0 \leq \beta_1^+ < \beta_1^+ < \cdots < \beta_n^+ < \beta_1^+ + 2\pi$ such that the equation mentioned above with some $C > 0$ and $\gamma \in \mathbb{R}$ are satisfied for all $k = 1, 2, \ldots, n$.

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Thank You !