A positivity conjecture related to the Riemann zeta function

Thomas Ransford
Université Laval, Quebec City, Canada

New Developments in Complex Analysis and Function Theory
University of Crete, July 2018

Collaborators

- Hugues Bellemare
- Yves Langlois

Article to appear in the *American Mathematical Monthly*.

The functions f_k

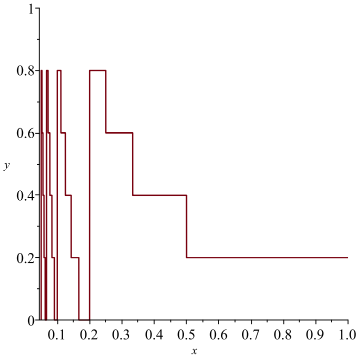
For each integer $k \geq 2$, define $f_k : (0,1) \to \mathbb{R}$ by

$$f_k(x) := \frac{1}{k}\left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{kx} \right\rfloor.$$

The functions f_k

For each integer $k \geq 2$, define $f_k : (0,1) \to \mathbb{R}$ by

$$f_k(x) := \frac{1}{k}\left\lfloor \frac{1}{x} \right\rfloor - \left\lfloor \frac{1}{kx} \right\rfloor.$$



Graph of $f_5(x)$

A least-squares problem

For $g : (0,1) \to \mathbb{C}$, we write

$$\|g\| := \Big(\int_0^1 |g(x)|^2 \, dx\Big)^{1/2}.$$

Set

$$d_n := \min_{\lambda_2,\dots,\lambda_n} \Big\| 1 - \sum_{k=2}^{n} \lambda_k f_k \Big\|.$$

Does $d_n \to 0$ as $n \to \infty$?

A least-squares problem

For $g : (0,1) \to \mathbb{C}$, we write

$$\|g\| := \left(\int_0^1 |g(x)|^2 \, dx \right)^{1/2}.$$

Set

$$d_n := \min_{\lambda_2,\ldots,\lambda_n} \left\| 1 - \sum_{k=2}^{n} \lambda_k f_k \right\|.$$

Does $d_n \to 0$ as $n \to \infty$?

Theorem (Nyman 1950, Báez-Duarte 2002)

$$d_n \to 0 \iff RH$$

The Riemann hypothesis

Euler (1748) : For $s > 1$,

$$\prod_{p} \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}.$$

The Riemann hypothesis

Euler (1748) : For $s > 1$,

$$\prod_p \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}.$$

Riemann (1859) : For s complex, $\operatorname{Re} s > 1$,

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = s \int_0^1 \left[\frac{1}{x}\right] x^{s-1}\, dx = \frac{s}{s-1} - s \int_0^1 \left\{\frac{1}{x}\right\} x^{s-1}\, dx.$$

The Riemann hypothesis

Euler (1748) : For $s > 1$,

$$\prod_p \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}.$$

Riemann (1859) : For s complex, $\operatorname{Re} s > 1$,

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} = s \int_0^1 \left[\frac{1}{x}\right] x^{s-1}\, dx = \frac{s}{s-1} - s \int_0^1 \left\{\frac{1}{x}\right\} x^{s-1}\, dx.$$

Riemann hypothesis

$\zeta(s) \neq 0$ for all s with $\operatorname{Re} s > 1/2$.

Proof that $d_n \to 0 \Rightarrow RH$

A simple calculation shows that, if $\operatorname{Re} s > 0$, then

$$\int_0^1 f_k(x)x^{s-1}\,dx = \frac{\zeta(s)}{s}\left(\frac{1}{k} - \frac{1}{k^s}\right).$$

Proof that $d_n \to 0 \Rightarrow RH$

A simple calculation shows that, if $\operatorname{Re} s > 0$, then

$$\int_0^1 f_k(x) x^{s-1}\, dx = \frac{\zeta(s)}{s}\left(\frac{1}{k} - \frac{1}{k^s}\right).$$

Suppose that $\zeta(s_0) = 0$, where $\operatorname{Re} s_0 > 1/2$. Then

$$\int_0^1 f_k(x) x^{s_0-1}\, dx = 0 \quad (k = 2, 3, \dots).$$

Proof that $d_n \to 0 \Rightarrow RH$

A simple calculation shows that, if $\operatorname{Re} s > 0$, then

$$\int_0^1 f_k(x) x^{s-1}\, dx = \frac{\zeta(s)}{s}\Big(\frac{1}{k} - \frac{1}{k^s}\Big).$$

Suppose that $\zeta(s_0) = 0$, where $\operatorname{Re} s_0 > 1/2$. Then

$$\int_0^1 f_k(x) x^{s_0-1}\, dx = 0 \quad (k = 2, 3, \dots).$$

Thus, for all $\lambda_2, \dots, \lambda_n$,

$$\int_0^1 \Big(1 - \sum_{k=2}^{n} \lambda_k f_k(x)\Big) x^{s_0-1}\, dx = \int_0^1 x^{s_0-1}\, dx = \frac{1}{s_0}.$$

Proof that $d_n \to 0 \Rightarrow RH$

A simple calculation shows that, if $\mathrm{Re}\, s > 0$, then

$$\int_0^1 f_k(x) x^{s-1}\, dx = \frac{\zeta(s)}{s}\Big(\frac{1}{k} - \frac{1}{k^s}\Big).$$

Suppose that $\zeta(s_0) = 0$, where $\mathrm{Re}\, s_0 > 1/2$. Then

$$\int_0^1 f_k(x) x^{s_0-1}\, dx = 0 \quad (k = 2, 3, \dots).$$

Thus, for all $\lambda_2, \dots, \lambda_n$,

$$\int_0^1 \Big(1 - \sum_{k=2}^n \lambda_k f_k(x)\Big) x^{s_0-1}\, dx = \int_0^1 x^{s_0-1}\, dx = \frac{1}{s_0}.$$

But also, by Cauchy–Schwarz,

$$\Big|\int_0^1 \Big(1 - \sum_{k=2}^n \lambda_k f_k(x)\Big) x^{s_0-1}\, dx\Big| \le \Big\|1 - \sum_{k=2}^n \lambda_k f_k\Big\|\, \|x^{s_0-1}\|.$$

Proof that $d_n \to 0 \Rightarrow RH$

A simple calculation shows that, if $\operatorname{Re} s > 0$, then

$$\int_0^1 f_k(x) x^{s-1}\, dx = \frac{\zeta(s)}{s}\Big(\frac{1}{k} - \frac{1}{k^s}\Big).$$

Suppose that $\zeta(s_0) = 0$, where $\operatorname{Re} s_0 > 1/2$. Then

$$\int_0^1 f_k(x) x^{s_0-1}\, dx = 0 \quad (k = 2, 3, \dots).$$

Thus, for all $\lambda_2, \dots, \lambda_n$,

$$\int_0^1 \Big(1 - \sum_{k=2}^{n} \lambda_k f_k(x)\Big) x^{s_0-1}\, dx = \int_0^1 x^{s_0-1}\, dx = \frac{1}{s_0}.$$

But also, by Cauchy–Schwarz,

$$\Big|\int_0^1 \Big(1 - \sum_{k=2}^{n} \lambda_k f_k(x)\Big) x^{s_0-1}\, dx\Big| \le \Big\|1 - \sum_{k=2}^{n} \lambda_k f_k\Big\| \, \|x^{s_0-1}\|.$$

Hence $d_n \ge 1/|s_0| \|x^{s_0-1}\|$ for all n. In particular $d_n \not\to 0$. $\square$

Behavior of (d_n) (Báez-Duarte et al, 2000)

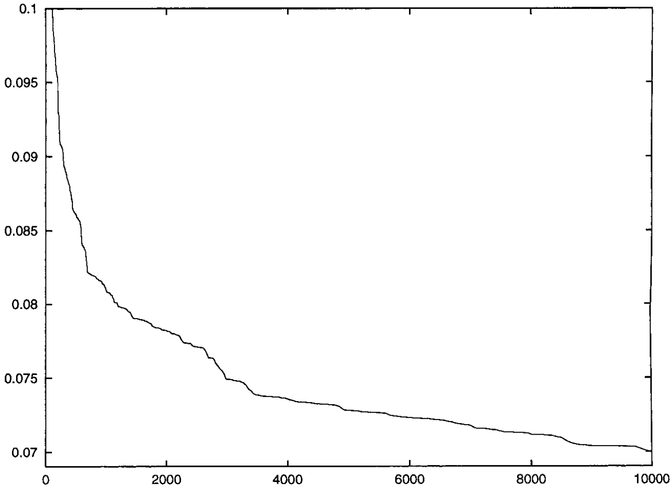


FIG 1. La distance d_n pour n de 2 à 10000.

Behavior of (d_n) (Baéz-Duarte et al, 2000)

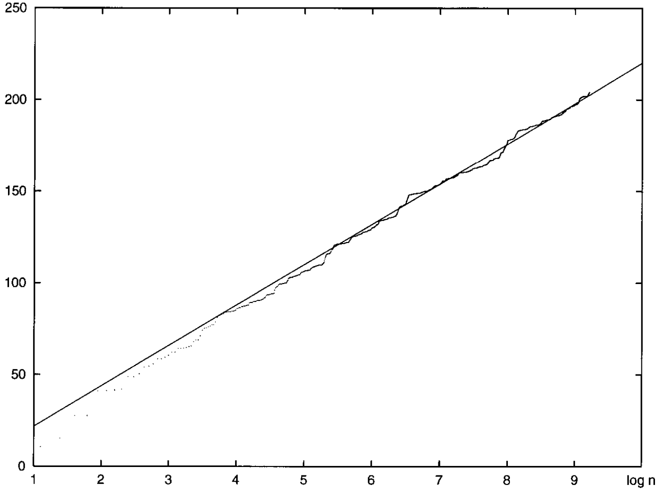


FIG. 2. Comparaison de d_n^{-2} et $21,99357 \log n$.

Behavior of the sequence (d_n)

Numerical calculations suggest that $d_n \approx 0.21/\sqrt{\log n}$.

Conjecture (Báez-Duarte, Balazard, Landreau, Saias (2000))

$$d_n \sim C_0/\sqrt{\log n}$$

where

$$C_0 := \sqrt{2 + \gamma - \log(4\pi)} = 0.2149\ldots$$

Theorem (Báez-Duarte, Balazard, Landreau, Saias (2000))

There exists $C > 0$ such that $d_n \geq C/\sqrt{\log n}$ for all n.

How to compute d_n ?

Notation : $\langle g, h \rangle := \int_0^1 g(x)h(x)\,dx$.

How to compute d_n ?

Notation : $\langle g, h \rangle := \int_0^1 g(x)h(x)\,dx$.

Gram–Schmidt : There is a unique sequence of functions $e_2, e_3, \ldots$ on $(0,1)$ such that :

- $\langle e_j, e_k \rangle = \delta_{jk}\ \forall j, k$,
- $\mathrm{span}\{e_2, \ldots, e_n\} = \mathrm{span}\{f_2, \ldots, f_n\}\ \forall n$,
- $\langle e_k, f_k \rangle > 0\ \forall k$.

How to compute d_n ?

Notation : $\langle g, h \rangle := \int_0^1 g(x)h(x)\,dx$.

Gram–Schmidt : There is a unique sequence of functions $e_2, e_3, \ldots$ on $(0,1)$ such that :

- $\langle e_j, e_k \rangle = \delta_{jk} \ \forall j, k$,
- $\operatorname{span}\{e_2, \ldots, e_n\} = \operatorname{span}\{f_2, \ldots, f_n\} \ \forall n$,
- $\langle e_k, f_k \rangle > 0 \ \forall k$.

A calculation shows that, for all choices of scalars $\lambda_2, \ldots, \lambda_n$,

$$\Big\|1 - \sum_{k=2}^{n} \lambda_k e_k\Big\|^2 = 1 - \sum_{k=2}^{n} |\langle 1, e_k \rangle|^2 + \sum_{k=2}^{n} |\lambda_k - \langle 1, e_k \rangle|^2.$$

How to compute d_n ?

Notation : $\langle g, h\rangle := \int_0^1 g(x)h(x)\,dx$.

Gram–Schmidt : There is a unique sequence of functions $e_2, e_3, \ldots$ on $(0,1)$ such that :

- $\langle e_j, e_k\rangle = \delta_{jk}\ \forall j, k$,
- $\operatorname{span}\{e_2, \ldots, e_n\} = \operatorname{span}\{f_2, \ldots, f_n\}\ \forall n$,
- $\langle e_k, f_k\rangle > 0\ \forall k$.

A calculation shows that, for all choices of scalars $\lambda_2, \ldots, \lambda_n$,

$$\Big\|1 - \sum_{k=2}^{n} \lambda_k e_k\Big\|^2 = 1 - \sum_{k=2}^{n} |\langle 1, e_k\rangle|^2 + \sum_{k=2}^{n} |\lambda_k - \langle 1, e_k\rangle|^2.$$

Distance formula

$$d_n^2 = 1 - \sum_{k=2}^{n} |\langle 1, e_k\rangle|^2.$$

How to compute $\langle 1, e_k \rangle$ in practice ?

Four steps :

- Compute $\langle 1, f_k \rangle$. This is easy since

$$\langle 1, f_k \rangle = \lim_{s \to 1} \int_0^1 f_k(x) x^{s-1}\, dx = \lim_{s \to 1} \frac{\zeta(s)}{s} \Big(\frac{1}{k} - \frac{1}{k^s} \Big) = \frac{\log k}{k}.$$

How to compute $\langle 1, e_k \rangle$ in practice ?

Four steps :

- Compute $\langle 1, f_k \rangle$. This is easy since

$$\langle 1, f_k \rangle = \lim_{s \to 1} \int_0^1 f_k(x) x^{s-1} \, dx = \lim_{s \to 1} \frac{\zeta(s)}{s} \left(\frac{1}{k} - \frac{1}{k^s} \right) = \frac{\log k}{k}.$$

- Compute $\langle f_j, f_k \rangle$ (see next slide).

How to compute $\langle 1, e_k \rangle$ in practice ?

Four steps :

- Compute $\langle 1, f_k \rangle$. This is easy since

$$\langle 1, f_k \rangle = \lim_{s\to 1} \int_0^1 f_k(x) x^{s-1}\, dx = \lim_{s\to 1} \frac{\zeta(s)}{s} \Big(\frac{1}{k} - \frac{1}{k^s} \Big) = \frac{\log k}{k}.$$

- Compute $\langle f_j, f_k \rangle$ (see next slide).
- Deduce $\langle e_j, f_k \rangle$ (see next slide).

How to compute $\langle 1, e_k \rangle$ in practice ?

Four steps :

- Compute $\langle 1, f_k \rangle$. This is easy since

$$\langle 1, f_k \rangle = \lim_{s \to 1} \int_0^1 f_k(x) x^{s-1} \, dx = \lim_{s \to 1} \frac{\zeta(s)}{s} \left(\frac{1}{k} - \frac{1}{k^s} \right) = \frac{\log k}{k}.$$

- Compute $\langle f_j, f_k \rangle$ (see next slide).
- Deduce $\langle e_j, f_k \rangle$ (see next slide).
- Compute $\langle 1, e_j \rangle$ by solving the triangular linear system

$$\sum_{j=2}^{n} \langle 1, e_j \rangle \langle e_j, f_k \rangle = \langle 1, f_k \rangle \quad (k = 2, \ldots, n).$$

The scalar products $\langle f_j, f_k \rangle$ and $\langle e_j, f_k \rangle$

To compute $\langle f_j, f_k \rangle$, we can use

Theorem (Vasyunin, 1996)

Let $j, k \geq 2$, and set $\omega := \exp(2\pi i/jk)$. Then

$$\langle f_j, f_k \rangle = \frac{1}{jk} \sum_{q=1}^{jk-1} \sum_{r=1}^{jk-1} \left\{ \frac{q}{j} \right\} \left\{ \frac{q}{k} \right\} \omega^{-qr} (\omega^{-r} - 1) \log(1 - \omega^r),$$

The scalar products $\langle f_j, f_k \rangle$ and $\langle e_j, f_k \rangle$

To compute $\langle f_j, f_k \rangle$, we can use

Theorem (Vasyunin, 1996)

Let $j, k \geq 2$, and set $\omega := \exp(2\pi i/jk)$. Then

$$\langle f_j, f_k \rangle = \frac{1}{jk} \sum_{q=1}^{jk-1} \sum_{r=1}^{jk-1} \left\{ \frac{q}{j} \right\} \left\{ \frac{q}{k} \right\} \omega^{-qr} (\omega^{-r} - 1) \log(1 - \omega^r),$$

And to obtain $\langle e_j, f_k \rangle$ we use

Proposition

Let $P_{jk} := \langle f_j, f_k \rangle$ and $L_{kj} := \langle f_k, e_j \rangle$. Then L is a lower-triangular matrix and $P = LL^t$ (Cholesky decomposition).

The matrix $P_{jk} := \langle f_j, f_k \rangle$ for $n = 9$

	2	3	4	5	6	7	8	9
2	0.1733	0.1063	0.1184	0.0918	0.0931	0.0784	0.0778	0.0683
3	0.1063	0.1770	0.1220	0.1118	0.1178	0.0976	0.0908	0.0914
4	0.1184	0.1220	0.1618	0.1194	0.1103	0.1023	0.1060	0.0912
5	0.0918	0.1118	0.1194	0.1456	0.1125	0.1019	0.0956	0.0918
6	0.0931	0.1178	0.1103	0.1125	0.1313	0.1049	0.0957	0.0909
7	0.0784	0.0976	0.1023	0.1019	0.1049	0.1192	0.0976	0.0889
8	0.0778	0.0908	0.1060	0.0956	0.0957	0.0976	0.1089	0.0910
9	0.0683	0.0914	0.0912	0.0918	0.0909	0.0889	0.0910	0.1002

The matrix $L_{kj} := \langle f_k, e_j \rangle$ for $n = 9$

	2	3	4	5	6	7	8	9
2	0.4163	0	0	0	0	0	0	0
3	0.2554	0.3343	0	0	0	0	0	0
4	0.2845	0.1475	0.2430	0	0	0	0	0
5	0.2205	0.1659	0.1325	0.2277	0	0	0	0
6	0.2237	0.1814	0.0819	0.0976	0.1792	0	0	0
7	0.1883	0.1480	0.1107	0.0929	0.0991	0.1764	0	0
8	0.1868	0.1288	0.1395	0.0638	0.0721	0.0841	0.1471	0
9	0.1641	0.1479	0.0934	0.0822	0.0651	0.0664	0.0863	0.1409

Is $\langle e_j, f_k \rangle$ always positive ?

Here is what we have been able to establish.

Is $\langle e_j, f_k \rangle$ always positive ?

Here is what we have been able to establish.

Numerical computation

$$\langle e_j, f_k \rangle > 0 \quad \text{for } 2 \leq j \leq k \leq 50000.$$

Is $\langle e_j, f_k \rangle$ always positive ?

Here is what we have been able to establish.

Numerical computation

$$\langle e_j, f_k \rangle > 0 \quad \text{for } 2 \leq j \leq k \leq 50000.$$

This leads us to formulate :

Conjecture

$$\langle e_j, f_k \rangle > 0 \quad \text{for all } j, k \text{ with } 2 \leq j \leq k.$$

Reformulation of the conjecture

It is possible to reformulate the conjecture purely in terms of the original functions (f_k), thanks to the following theorem.

Theorem

$$\langle e_j, f_k \rangle > 0 \iff \begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & \langle f_2, f_k \rangle \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & \langle f_3, f_k \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & \langle f_j, f_k \rangle \end{vmatrix} > 0.$$

Reformulation of the conjecture

It is possible to reformulate the conjecture purely in terms of the original functions (f_k), thanks to the following theorem.

Theorem

$$\langle e_j, f_k \rangle > 0 \iff \begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & \langle f_2, f_k \rangle \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & \langle f_3, f_k \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & \langle f_j, f_k \rangle \end{vmatrix} > 0.$$

Remark : The $\langle f_j, f_k \rangle$ are directly related to the zeta-function by

$$\langle f_j, f_k \rangle = \frac{1}{2\pi jk} \int_{-\infty}^{\infty} \left(j^{\frac{1}{2}-it} - 1\right)\left(k^{\frac{1}{2}+it} - 1\right) \frac{\left|\zeta(\frac{1}{2} + it)\right|^2}{\frac{1}{4} + t^2} \, dt.$$

Asymptotic formula for $\langle f_j, f_k \rangle$

Theorem (based on Báez-Duarte, Balazard, Landreau, Saias, 2005)

For each $j \geq 2$,

$$\lim_{k \to \infty} \frac{\langle f_j, f_k \rangle}{(\log k)/2k} = \frac{j-1}{j}.$$

Asymptotic formula for $\langle f_j, f_k \rangle$

Theorem (based on Báez-Duarte, Balazard, Landreau, Saias, 2005)

For each $j \geq 2$,

$$\lim_{k\to\infty} \frac{\langle f_j, f_k \rangle}{(\log k)/2k} = \frac{j-1}{j}.$$

Corollary

Let $j \geq 2$. *If*

$$\begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & 1/2 \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & 2/3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & (j-1)/j \end{vmatrix} > 0,$$

then there exists $k_0(j)$ *such that* $\langle f_k, e_j \rangle > 0$ *for all* $k \geq k_0(j)$.

Asymptotic formula for $\langle f_j, f_k \rangle$

Theorem (based on Báez-Duarte, Balazard, Landreau, Saias, 2005)

For each $j \geq 2$,

$$\lim_{k\to\infty} \frac{\langle f_j, f_k \rangle}{(\log k)/2k} = \frac{j-1}{j}.$$

Corollary

Let $j \geq 2$. If

$$\begin{vmatrix} \langle f_2, f_2 \rangle & \langle f_2, f_3 \rangle & \dots & \langle f_2, f_{j-1} \rangle & 1/2 \\ \langle f_3, f_2 \rangle & \langle f_3, f_3 \rangle & \dots & \langle f_3, f_{j-1} \rangle & 2/3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle f_j, f_2 \rangle & \langle f_j, f_3 \rangle & \dots & \langle f_j, f_{j-1} \rangle & (j-1)/j \end{vmatrix} > 0,$$

then there exists $k_0(j)$ such that $\langle f_k, e_j \rangle > 0$ for all $k \geq k_0(j)$.

Remark. The hypothesis of the corollary holds for $2 \leq j \leq 100$.

Conclusion

Questions :

- Is the conjecture true ?

Conclusion

Questions :

- Is the conjecture true ?
- If yes, does it imply RH ?

Conclusion

Questions :

- Is the conjecture true ?
- If yes, does it imply RH ?
- Perhaps it is a consequence of RH ?

Conclusion

Questions :

- Is the conjecture true ?
- If yes, does it imply RH ?
- Perhaps it is a consequence of RH ?

All these questions remain open...

Conclusion

Questions :

- Is the conjecture true ?
- If yes, does it imply RH ?
- Perhaps it is a consequence of RH ?

All these questions remain open...

THANK YOU !