Approximation of functions and operators. A tentative comparison

H. Queffélec, Univ. of Lille 1
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Joint work with D. Li, L. Rodríguez-Piazza

To commute or not to commute

Functions	Operators
$K = [-1, 1]$	H, Hilbert space
$f \in L^\infty(K)$ with $\|f\|_K = \cdots$	$T \in \mathcal{L}(H)$ with $\|T\| = \cdots$
$\mathcal{P}_n$: polynomials of degree $\leq n$	$\mathcal{R}_n$: operators of rank $< n$.
$E_n(f) = \inf_{P \in \mathcal{P}_n} \|f - P\|_K$	$a_n(T) = \inf_{R \in \mathcal{R}_n} \|T - R\|$.
$f \in \mathcal{C}(K) \Leftrightarrow E_n(f) \to 0$	$T \in \mathcal{K}(H) \Leftrightarrow a_n(T) \to 0$
$E_n(f) = \varepsilon_n \downarrow 0$ arbitrary (Bernstein)	$a_n(T) = \varepsilon_n \downarrow 0$ arbitrary (trivial)
$E_n(f)$ small iff f regular	$a_n(T)$ small : $T \in S_p, p > 0$.
Green capacity of $K \subset \Omega$	Green capacity of $\varphi(\mathbb{D}) \subset \mathbb{D}$
Bernstein-Widom formula	Li-Q-Rodr. spectral radius formula

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$\hookrightarrow$ What in item 6 if $T \in \mathcal{C}$? (Hankel, composition)

Approximation of functions

Let $0 < r < 1$. Set

$$K = [-1, 1] \subset \Omega = \Omega_r = \{z : |z - 1| + |z + 1| < r + r^{-1}\}.$$

Ω_r is the interior of an ellipse.

Theorem (S. Bernstein 1912)

Let $f \in \mathcal{C}(K)$. The following "spectral radius formula" holds true

$$\limsup_{n \to \infty} [E_n(f)]^{1/n} \leq r \iff f \text{ extends analytically to } \Omega = \Omega_r.$$

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Key facts :

$$P \in \mathcal{P}_n,\ z \notin K \Rightarrow |P(z)| \leq \|P\|_K \, |z + \sqrt{z^2 - 1}|^n$$

$$\Omega = K \cup \{z : |z + \sqrt{z^2 - 1}| < r^{-1}\}.$$

Bernstein, variant

To ease the presentation, we take another model, in which Ω is fixed and K becomes variable. We denote

$$\mathbb{D} = \{|z| < 1\},\ \mathbb{T} = \{z : |z| = 1\} \text{ and } K_r = r\mathbb{T}.$$

One has $(K, \Omega_r) = \big(J(K_r), J(\mathbb{D})\big)$, where $J(w) = \frac{r^{-1}w + rw^{-1}}{2}$ is the Joukovski map.

Bernstein's theorem can be rephrased, with $E_n(f) = \inf_{P \in \mathcal{P}_n} \|f - P\|_{K_r}$:

Theorem (Bernstein)

Let $f \in \mathcal{C}(K_r)$. Then

$$\limsup_{n\to\infty} [E_n(f)]^{1/n} \leq r \iff f \text{ extends analytically to } \mathbb{D}.$$

We now need a small detour...

Geometric meaning

Indeed, r is related to the Green capacity of K_r inside $\mathbb{D}$.

We define the Green capacity $C_1(K)$ of $K \subset \mathbb{D}$ compact :

$$C_1(K) = \sup_{0 \leq u \leq 1} \int_K (\Delta u)(z) \frac{dxdy}{2\pi}, \quad u \text{ subharmonic, or}$$

$$C_1(K) = C_1(\partial K) = \sup\{\mu(K) : G_\mu(z) \leq 1 \text{ for all } \ z \in \mathbb{D}\}$$

where $G_\mu(z) = \int_{\mathbb{D}} \log \left| \frac{1-\overline{\zeta} z}{z-\zeta} \right| d\mu(\zeta)$ is the Green potential of μ.

We also set $\quad \Gamma_1(K) = \exp\left[-1/(C_1(K)) \right].$

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Example : Let $r < 1$. Then

$$C_1(r\mathbb{D}) = C_1(r\mathbb{T}) = \frac{1}{\log(1/r)} \text{ and } \Gamma_1(r\mathbb{D}) = r.$$

Generalisations

Under this form, Walsh extended Bernstein's result to arbitrary compact sets $K \subset \mathbb{D}$.

Theorem (Bernstein-Walsh)

For $f \in \mathcal{C}(K)$ and $K \subset \mathbb{D}$ compact :

$$\limsup_{n\to\infty} [E_n(f)]^{1/n} \leq \Gamma_1(K) \iff f \text{ extends analytically to } \mathbb{D}.$$

And Siciak extended Bernstein-Walsh to dimension d.

Theorem (Siciak)

For $f \in \mathcal{C}(K)$ and $K \subset \mathbb{D}^d$ compact :

$$\limsup_{n\to\infty} [E_n(f)]^{1/n} \leq \Gamma_d(K) \iff f \text{ extends analytically to } \mathbb{D}^d.$$

Observe that, now, $\dim\ \mathcal{P}_n \approx n^d$.

A result of Widom on Operators

We now switch to operators, and use other subspaces than $\mathcal{P}_n$.

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$, let

$$J = J_K : H^\infty(\mathbb{D}) \to \mathcal{C}(K),\ Jf = f$$

be the canonical injection, and

$$\delta_n(J) = \inf_{\dim E < n} \Big[\sup_{f \in B_{H^\infty(\mathbb{D})}} d(Jf, E) \Big]$$

where $d(f, E) = \inf_{P \in E} \|f - P\|_K$. The number $\delta_n(J)$ is called the n-th Kolmogorov number of J.

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Theorem (Widom)

Let $K \subset \mathbb{D}$ compact with non-empty interior. Then

$$\limsup_{n \to \infty} [\delta_n(J)]^{1/n} = \Gamma_1(K).$$

Approximation of operators

Let $T : H \to H$ be a bounded operator.

We recall that its n-th approximation number $a_n(T)$ is

$$a_n = a_n(T) = \inf_{R \in \mathcal{R}_n} \|T - R\|.$$

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3. Rate of decay of a_n. For example, $a_n \approx e^{-\sqrt{n}}$?

$\hookrightarrow$ We focus on the class of composition operators C_φ.

Composition operators

We set $\Omega = \mathbb{D}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. The Hardy space is

$$H = H^2(\Omega) = \left\{ f(z) = \sum_\alpha b_\alpha z^\alpha ; \sum_\alpha |b_\alpha|^2 =: \|f\|^2 < \infty \right\}.$$

The monomials (z^α) form an orthonormal basis of H.

If now $\varphi : \Omega \to \Omega$ is analytic and $C_\varphi(f) = f \circ \varphi$, then $C_\varphi : H \to Hol(\Omega)$.

The question is :

1. When does C_φ map H to itself ?

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The question is :

1. When does C_φ map H to itself ?
2. Compare the operator C_φ and its symbol φ, as one does for Hankel operators.

Classics for $d = 1$

We set $T = C_\varphi$ with φ non-constant. Then :

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3. T p-Schatten characterized (Luecking, Zhu).

More recent results for $d = 1$

The study of the decay rate of $a_n(T)$, $T = C_\varphi$ compact, was undertaken in 2012 (Li-Q-Rodríguez-Piazza).

Motivation : perform a study parallel to that of compact Hankel operators H_φ, shown to have arbitrary approximation numbers $\varepsilon_n = a_n(H_\varphi) \downarrow 0$. (Megretski-Peller-Treil (1995) and Gérard-Grellier (2014)).

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4. $\|\varphi\|_\infty = 1 \Rightarrow a_n(T) \geq \delta e^{-n\varepsilon_n}$ with $\varepsilon_n \to 0$.

Spectral radius formula

One of our main results later (2015) was, in case $K = \overline{\varphi(\mathbb{D})} \subset \mathbb{D}$:

$$\beta_1(T) := \lim_{n\to\infty} \left[a_n(T)\right]^{1/n} = \Gamma_1(K).$$

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2. Fact : $0 \in K_j$, connected, *and* $\lim_{j\to\infty} |K_j| = 1 \Longrightarrow C_1(K_j) \to \infty$.
3. Take $K_j = \overline{\varphi(r_j\mathbb{D})},\ r_j \uparrow 1$.

Test : $\varphi(z) = rz,\ a_n(C_\varphi) = r^{n-1},\ K = r\overline{\mathbb{D}},\ \beta_1(C_\varphi) = r$.

We also know that $\Gamma_1(K) = r$ since $C_1(K) = \frac{1}{\log 1/r}$.

Results for $d > 1$

Let $\varphi : \mathbb{D}^d \to \mathbb{D}^d$ be analytic and non-degenerate, i.e.$J_\varphi \not\equiv 0$, and

$$C_\varphi(f) = f \circ \varphi : H \; to \; \mathrm{Hol}(\mathbb{D}^d).$$

Difficulty : C_φ is not always bounded on H !

We obtained the following when $T = C_\varphi$ is bounded, where this time :

$$\beta_d(T) := \lim_{n\to\infty} \left[a_{n^d}(T)\right]^{1/n},$$

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4. $\|\varphi\|_\infty = 1 \Rightarrow a_{n^d}(T) \geq \delta e^{-n\varepsilon_n}$ with $\varepsilon_n \to 0$, or else $\beta_d(C_\varphi) = 1$, in case φ separates variables.

We will come back to the general case.

A word on pluricapacity

The following was coined by Bedford and Taylor (around 1980).

1. Replace subharmonic by plurisubharmonic.
2. Replace the laplacian (trace) by the Monge-Ampère operator (determinant).

You get the pluri, or Bedford-Taylor, capacity $C_d(K)$ of $K \subset \Omega = \mathbb{D}^d$.
This new parameter verifies (Blocki) :

$$C_d(K_1 \times \cdots \times K_r) = \prod_{j=1}^{r} C_1(K_j), \text{ where } K_j \subset \mathbb{D}.$$

In particular, it extends the Green capacity to dimension d.

A conjecture of Kolmogorov

The following was conjectured by Kolmogorov, and proved later.

Theorem (Nivoche-Zaharyuta)

Let $K \subset \mathbb{D}^d$ be compact, with non-void interior, and "regular". Set

$$\Gamma_d(K) = \exp\left[-\left(\frac{d!}{C_d(K)}\right)^{1/d}\right]$$

and let $J : H^\infty(\mathbb{D}^d) \to \mathcal{C}(K)$ be the canonical injection. Then

$$\limsup_{n\to\infty} \left[\delta_{n^d}(J)\right]^{1/n} = \Gamma_d(K).$$

We now examine a simple example.

An example

Let $\varphi(z) = (r_1 z_1, \dots, r_d z_d)$ with $0 < r_j < 1$. Set $\rho_j = \log 1/r_j$.

So that $K := \overline{\varphi(\mathbb{D}^d)} = \prod_{j=1}^{d} r_j \overline{\mathbb{D}}$ and, by Blocki and the definition :

$$C_d(K) = \frac{1}{\rho_1 \cdots \rho_d},$$

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We now see that $C_\varphi(z^\alpha) = r^\alpha z^\alpha$ if $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$.

Hence, the $a_n(C_\varphi)$ are the decreasing rearrangement of the numbers $r_1^{\alpha_1} \cdots r_d^{\alpha_d}$.

An example, continued

Let

$$N_A = |\{\alpha : \sum_{j=1}^{d} \alpha_j \rho_j \leq A\}| \text{ with } \rho_j = \log 1/r_j.$$

Then $N_A = |\{\alpha : r^\alpha \geq e^{-A}\}|$ and

$$N_A \sim \frac{A^d}{d!\rho_1 \cdots \rho_d} \text{ as } A \to \infty.$$

An example, continued

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$$N_A \sim \frac{A^d}{d!\rho_1 \cdots \rho_d} \text{ as } A \to \infty.$$

If $A = \log 1/a_N$, then $N_A = N$. Take $N = n^d$ and invert to get

$$a_{n^d}(C_\varphi) = \exp\Big[- n(1 + o(1))\big(d!\rho_1 \cdots \rho_d\big)^{1/d}\Big],$$

implying the spectral radius formula, with $K = \overline{\varphi(\mathbb{D}^d)}$, namely :

$$\beta_d(C_\varphi) := \lim_{n\to\infty} \big(a_{n^d}(C_\varphi)\big)^{1/n} = \Gamma_d(K).$$

Spectral formula for $d > 1$?

This seems to mean that the Bedford-Taylor capacity is the right substitute to Green capacity in dimension $d > 1$ and will prove as useful for the study of composition operators. The truth so far is

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1. The spectral radius formula essentially holds in several dimensions.
2. But it is possibly less useful, even though it provides a very simple proof of item 3 in slide thirteen.
3. Indeed, some results are simply wrong in dimension $d > 1$!

A counterexample

Our main result (counterexample) states

Theorem

There exists a family of maps $\varphi : \mathbb{D}^2 \to \mathbb{D}^2$ *such that*

1. *The family contains an injective map.*
2. *The operator* $C_\varphi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ *is compact.*
3. *It holds* $\|\varphi\|_\infty = 1$ *and* φ *is non-degenerate, so* $\beta_2^-(C_\varphi) > 0$.
4. *The approximation numbers satisfy* $a_{n^2}(C_\varphi) \leq \alpha e^{-\beta n}$, *i.e.* $\beta_2^+(C_\varphi) < 1$.

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2. *The operator $C_\varphi : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is compact.*
3. *It holds $\|\varphi\|_\infty = 1$ and φ is non-degenerate, so $\beta_2^-(C_\varphi) > 0$.*
4. *The approximation numbers satisfy $a_{n^2}(C_\varphi) \leq \alpha e^{-\beta n}$, i.e. $\beta_2^+(C_\varphi) < 1$.*

The proof combines

1. Rudin functions ($\langle h^p, h^q \rangle = 0$ if $p \neq q$)

A counterexample

Our main result (counterexample) states

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The proof combines

1. Rudin functions ($\langle h^p, h^q \rangle = 0$ if $p \neq q$)
2. Weighted composition operators $M_w C_\varphi$ and their approximation numbers
3. Hilbertian sums of operators ($T = \oplus_{k \geq 0} T_k$).

Rudin functions

Those are functions $h \in B_{H^\infty}$ all of whose powers h^p are orthogonal in H^2, e.g. an inner function with $h(0) = 0$, but there are others (Bishop). Then, we take the triangular symbol

$$\varphi(z_1, z_2) = (\lambda(z_1), \mu(z_1)h(z_2)).$$

First, if $f(z) = \sum_{j,k} c_{j,k} z_1^j z_2^k$, then

$$f(z) = \sum_{k\geq 0} z_2^k f_k(z_1) \text{ where } f_k(z_1) = \sum_j c_{j,k} z_1^j$$

and by orthogonality

$$\|f\|^2 = \sum_{k\geq 0} \|f_k\|^2_{H^2(\mathbb{D})}$$

Weighted composition operators

$M_w =$ multiplication operator $f \mapsto wf : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, $w \in H^\infty$.
The $a_n(M_w C_\varphi)$ were studied independently ([3]). Now,

$$C_\varphi f(z) = \sum_{k\geq 0} (h(z_2))^k \left(\mu(z_1)^k \sum_j c_{j,k} \lambda(z_1)^j \right)$$

$$= \sum_{k\geq 0} (h(z_2))^k \left(M_{\mu^k} C_\lambda f_k(z_1) \right)$$

so that, by Rudin orthogonality

$$\| C_\varphi f \|^2 \leq \sum_{k\geq 0} \| T_k f_k \|^2$$

where T_k is the weighted composition operator

$$T_k = M_{\mu^k} C_\lambda .$$

Hilbertian sums

We can hence assume that

$$T = \oplus_{k\geq 0} T_k \text{ where } T_k = M_{\mu^k} C_\lambda$$

and have the simple

Lemma

If $T = \oplus_{k\geq 0} T_k$ *and* $N = n_0 + \cdots + n_r$, *then*

$$a_N(T) \leq \max\Big(\sum_{k=0}^{r} a_{n_k}(T_k),\ \sup_{k>r} \|T_k\|\Big).$$

Our choice is

1. $\lambda = \frac{1+\lambda_\theta}{2}$ where λ_θ is a lens map with $0 < \theta < 1$.
2. $\mu = w \circ \lambda$ where $w(z) = \exp\left[-\left(\frac{1+z}{1-z}\right)^\theta\right]$. One has $\|\mu\|_\infty < 1$.

End of proof

We finish with the following simple key ! lemma

Lemma

Let $n_k = n\mu_k = n\log(n/k)$. Then, for some constants $a, b > 0$:

1. $a_{n_k}(T_k) \leq ae^{-bn}$ *for* $1 \leq k \leq n$.
2. $\|T_k\| \leq e^{-bk}$ *for* $k > n$.
3. $\sum_{k=1}^{n} n_k = n\sum_{k=1}^{n}(\log n/k) \lesssim n^2$.

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Combining the previous lemmas gives the result. As soon as h is injective (e.g. $h(z) = z$), φ is injective as well.

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THANKS FOR YOUR ATTENTION!