Quasiregularly Elliptic Manifolds and Cohomology

Eden Prywes

University of California, Los Angeles

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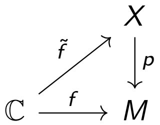
Let M be a Riemann surface and

$$f : \mathbb{C} \to M$$

a nonconstant holomorphic map.

- What type of surface can M be?
    - By the uniformization theorem, the universal cover X of M is $\mathbb{D}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$.

$$\begin{array}{ccc} & & X \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow p \\ \mathbb{C} & \xrightarrow{f} & M \end{array}$$



- If $X = \mathbb{D}$, then f is constant.
- If $X = \widehat{\mathbb{C}}$, then $M = \widehat{\mathbb{C}}$.
- If $X = \mathbb{C}$, then $M \simeq S^1 \times S^1$.

How can we generalize this to higher dimensions?
conformal $\rightarrow$ quasiconformal
holomorphic $\rightarrow$ quasiregular

Quasiregular Maps

Let M be a closed, connected, orientable Riemannian manifold.

Defintion

A map $f\colon \mathbb{R}^n \to M$ is K-*quasiregular* if $f \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$, f is nonconstant and

$$||Df||^n \leq KJ_f$$

- A homeomorphic K-quasiregular map is K-quasiconformal.
- A 1-quasiregular map in dimension 2 is holomorphic.

Question

What manifolds admit quasiregular maps (quasiregularly elliptic)?

Revisit $\mathbb{C}$

A quasiregular map $f\colon \mathbb{C} \to M$ can always be decomposed

$$f = g \circ \phi$$

where $\phi\colon \mathbb{C} \to \mathbb{C}$ is quasiconformal and $g\colon \mathbb{C} \to M$ is holomorphic (Stoïlow's theorem).
So in dimension 2 the question of quasiregular ellipticity reduces to the holomorphic case.

Fundamental Group

In dimension 2, the fundamental group was the main obstruction for admitting holomorphic maps.

Theorem (Varopoulos)

If M is an n-dimensional Riemannian manifold that is quasiregularly elliptic, then $\pi_1(M)$ has a growth order bounded above by n.

- Proof relies on lifting f to a noncompact universal covering space.
- As in dimension 2, this result is independent of the distortion K.
- Gromov ('81) asked whether there exists a simply connected manifold that is not quasiregularly elliptic.

K-Dependency

The situation is not identical for $K = 1$ and $K > 1$.

Theorem (Rickman '80)

A K-quasiregular map $f : \mathbb{R}^n \to S^n$ can omit at most $C(n, K)$-points.

Theorem (Rickman '85, Drasin and Pankka '15)

For $N \in \mathbb{N}$, there exists a quasiregular map $f : \mathbb{R}^n \to S^n$ that omits N points.

- In higher dimensions, the distortion constant can lead to different results.

K-Dependency

We can look for obstructions in other invariants besides the fundamental group.

Theorem (Bonk and Heinonen '01)

If M is K-quasiregularly elliptic, then

$$\dim H^l(M) \leq C(n, l, K),$$

where $H^l(M)$ is the degree l de Rham cohomology of M.

They conjecture that $C(n, l, K) = \binom{n}{l}$, which is attained since T^n is quasiregularly elliptic.

Dynamic Result

Theorem (Kangasniemi '17)

If M admits a noninjective uniformly quasiregular map, then

$$\dim H^l(M) \leq \binom{n}{l}.$$

- A result by Martin, Volker and Peltonen ('06) gives that M is quasiregularly elliptic.
- Proof uses pointwise orthogonality properties of rescaled differential forms on M.

Main Result

What about the case when M is not assumed to admit a uniformly quasiregular map?

Theorem (P. '18)

If M is K-quasiregularly elliptic, then

$$\dim H^l(M) \leq \binom{n}{l}$$

- This bound is optimal because T^n is quasiregularly elliptic.

Main Result

Corollary (P. '18)

There exist simply connected manifolds that are not quasiregularly elliptic.

- For example, $M = \#^m(S^2 \times S^2)$ for $m \geq 4$.

Theorem (Rickman '06)

$(S^2 \times S^2)\#(S^2 \times S^2)$ is quasiregularly elliptic.

Outline of the Proof

- Using f, pull back Poincaré pairs on M.
- We then rescale the forms in $\mathbb{R}^n$ to get a collection of differential forms on $B(0,1)$
- Lastly, we show that the rescaled forms are pointwise orthogonal, which says that the number of forms should be bounded above by $\dim \bigwedge^l \mathbb{R}^n = \binom{n}{l}$.
  - This uses a weak reverse Hölder inequality for Jacobians of quasiregular maps into manifolds with nontrivial cohomology.

Rescaling Procedure

In the proof of the Bonk and Heinonen result the authors use a rescaling procedure on the map $f : \mathbb{R}^n \to M$.

- This gives that f is uniformly Hölder continuous.

Instead of rescaling the map f, rescale the pullbacks of differential forms.
Rescaling functions in the Rickman-Picard theorem context was used in a paper by Eremenko and Lewis '91.

- They rescale $\mathcal{A}$-harmonic functions of the form $\log |f|$ with a similar normalization to get functions on $B(0,1)$.
- The new functions satisfy strong pointwise estimates.

Orthogonality

If $k = \dim H^l(M)$, then, on M, let $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ be Poincaré pairs.

$$\int_M \alpha_a \wedge \beta_b = \delta_{ab}$$

So, if $\eta_a = f^*\alpha_a$ and $\theta_b = f^*\beta_b$, then in the rescaling

$$\tilde{\eta}_a \wedge \tilde{\theta}_b = 0$$

$a \neq b$, for almost every $x \in B(0,1)$.
At each point there can only be $\binom{n}{l}$ nonzero differential forms.
Equidistribution properties of f lead to a contradiction.

Reverse Hölder Inequality I

In the argument above actually need to use a reverse Hölder inequality for J_f.

Theorem (Bojarski and Iwaniec '83)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiregular map. Then $f \in W^{1,nq}_{loc}(\mathbb{R}^n)$ for $1 < q \leq Q(n, K)$, where $Q(n, K)$ depends only on n and K. If $B \subset \mathbb{R}^n$ is a ball, then

$$\left(\int_{\frac{1}{2}B} J_f^q \right)^{1/q} \leq C(n, q, K) \frac{1}{|B|^{1/q'}} \int_B J_f \tag{1}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Crucially, $C(n, q, K)$ is independent of f and B.

- This theorem does not directly apply since $f : \mathbb{R}^n \to M$. If $H^l(M) = 0$ for $1 \leq l \leq n - 1$, then the theorem does not necessarily hold.

Reverse Hölder Inequality II

In our case there is an l so that $H^l(M) \neq 0$.

Proposition

Let M be a closed Riemannian manifold and let $f : \mathbb{R}^n \to M$ be K-quasiregular. If there exists an integer l with $1 \leq l \leq n-1$ such that $H^l(M) \neq 0$, then the Jacobian of f satisfies the weak reverse Hölder inequality,

$$\frac{1}{|\frac{1}{2}B|} \int_{\frac{1}{2}B} J_f \leq C(n,M,K) \left(\frac{1}{|B|} \int_B J_f^{n/(n+1)} \right)^{(n+1)/n},$$

where $B \subset \mathbb{R}^n$ is an arbitrary ball.

- Once the proposition is shown, then the reverse Hölder inequality for an exponent $b > 1$ follows from Gehring's lemma.

Further questions

- What about the case when M is not compact?
  - For $n = 2$, $M \simeq \mathbb{C}$ or $S^1 \times \mathbb{R}$.
  - For $n > 2$, the answer must depend on K by the Rickman-Picard theorem.
- Does there exist a quasiregularly elliptic manifold where the quasiregular map does not factor through the torus?
  - If $\#^3 S^2 \times S^2$ is quasiregularly elliptic, then the map cannot factor through the torus (Pankka and Souto '12).
- Suppose $\dim H^l(M) = \binom{n}{l}$, what does this imply about M?
  - For $l = 1$, there must exist a covering map $p\colon T^n \to M$ (Luisto and Pankka '16).

Thank you!