# Strong continuity of semigroups of composition operators on Morrey spaces

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## Spaces of analytic functions in the unit disc

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.
$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

## Automorphisms on $\mathbb{D}$

We consider

$$Aut(\mathbb{D}) = \{\varphi : \mathbb{D} \to \mathbb{D} : \varphi \text{ is conformal}\}.$$

It is known that

$$Aut(\mathbb{D}) = \{\lambda\sigma_a : |\lambda| = 1, \quad a \in \mathbb{D}\}$$

where $\sigma_a : \mathbb{D} \to \mathbb{D}$ is the Möbius map $\sigma_a(z) = \frac{z-a}{1-\bar{a}z}$.

## Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

If $0 < p \leq \infty$, we consider the Hardy spaces $H^p$,

$$H^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) \, : \, \|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty \right\}.$$

## Bergman spaces

If $0 < p < \infty$, we consider the Bergman spaces $A^p$,

$$A^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) \, : \, \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty \right\}.$$

## BMOA

$$BMOA = \left\{ f \in H^1 : f\left(e^{i\theta}\right) \in BMO \right\}.$$

$$H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p.$$

## Bloch space

$$\mathcal{B} = \{f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

## Morrey spaces

For $0 < \lambda < 1$ we define the Morrey space $\mathcal{L}^{2,\lambda}$ as

$$\mathcal{L}^{2,\lambda} = \left\{ f \in H^2 \; : \; \sup_{a \in \mathbb{D}} (1 - |a|^2)^{\frac{1-\lambda}{2}} \| f \circ \sigma_a - f(a) \|_{H^2} < \infty \right\}.$$

We also define for $0 < \lambda < 1$ the *little* Morrey spaces $\mathcal{L}_0^{2,\lambda}$ as

$$\mathcal{L}_0^{2,\lambda} = \left\{ f \in \mathcal{L}^{2,\lambda} \; : \; \lim_{|a| \to 1} (1 - |a|^2)^{\frac{1-\lambda}{2}} \| f \circ \sigma_a - f(a) \|_{H^2} = 0 \right\}.$$

For $0 < \lambda < 1$

$$BMOA = \mathcal{L}^{2,1} \subset \mathcal{L}^{2,\lambda} \subset \mathcal{L}^{2,0} = H^2.$$

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \in \mathcal{B} \setminus \mathcal{L}^{2,\lambda} \qquad 0 < \lambda < 1.$$

$$f(z) = (1-z)^{-\frac{1-\lambda}{2}} \in \mathcal{L}^{2,\lambda} \setminus \mathcal{B} \quad 0 < \lambda < 1.$$

## Growth in Morrey spaces

For $0 < \lambda < 1$ there exists a constant $C$ such that if $f \in \mathcal{L}^{2,\lambda}$ then

$$|f(z)| \leq \frac{C}{(1-|z|)^{\frac{1-\lambda}{2}}} \quad z \in \mathbb{D}.$$

It follows that

$$\mathcal{L}^{2,\lambda} \subset \mathcal{B}^{\frac{3-\lambda}{2}} \quad 0 < \lambda < 1.$$

## $\alpha$-Bloch spaces

If $\alpha > 0$ we can consider the spaces

$$\mathcal{B}^{\alpha} = \{f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha}|f'(z)| < \infty\}.$$

$$\mathcal{B} = \mathcal{B}^1 \subset \mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}, \quad 1 \leq \alpha_1 \leq \alpha_2.$$

## Semigroups of analytic functions

A semigroup $(\varphi_t)$ for $t \geq 0$ consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ which satisfies the following:

- $\varphi_0$ is the identity in $\mathbb{D}$.
- $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$.
- $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

## Semigroups of composition operators

Each semigroup $(\varphi_t)$ gives rise to a semigroup $(C_t)$ consisting on composition operators on $\mathcal{H}ol(\mathbb{D})$,

$$C_t(f) = f \circ \varphi_t, \quad f \in \mathcal{H}ol(\mathbb{D}).$$

We are going to be interested in the restriction of $(C_t)$ to certain linear subspaces of $\mathcal{H}ol(\mathbb{D})$.

**Definition**

Given a Banach space $X \subset \mathcal{H}ol(\mathbb{D})$ and a semigroup $(\varphi_t)$, we say that $(\varphi_t)$ generates a semigroup of operators on $X$ if $(C_t)$ is a well defined **strongly continuous** semigroup of bounded operators in $X$.

This means that for every $f \in X$, we have $C_t(f) \in X$ for all $t \geq 0$ and

$$\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0.$$

## Definition

For a semigroup $(\varphi_t)$ we define the **infinitesimal generator** $G$ of $(\varphi_t)$ as

$$G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t}, \quad z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$ so $G \in \mathcal{H}ol(\mathbb{D})$. Moreover

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z}, \; z \in \mathbb{D}, \; t \geq 0.$$

## Examples of semigroups

Some examples of semigroups are:

- $\varphi_t(z) = z,\ t \geq 0 \quad G(z) = 0$ (Trivial semigroup).
- $\varphi_t(z) = e^{-t}z,\ t \geq 0 \quad G(z) = -z.$
- $\varphi_t(z) = e^{it}z,\ t \geq 0 \quad G(z) = iz.$

## Representation of the infinitesimal generator

$G$ has a unique representation

$$G(z) = (\overline{b}z - 1)(z - b)P(z),\ z \in \mathbb{D},$$

where $b \in \overline{\mathbb{D}}$ and $P \in \mathcal{H}ol(\mathbb{D})$ with $Re\, P(z) \geq 0$ for all $z \in \mathbb{D}$.
If $G \not\equiv 0$, $(b, P)$ is uniquely determined from $(\varphi_t)$.

The point $b$ is called **Denjoy-Wolff point** of the semigroup.

## Denjoy-Wolff point in the disc

Studying the semigroup in the case $b \in \mathbb{D}$ can be reduced by renormalization to the case $b = 0$. Then

$$\varphi_t(z) = h^{-1}\left(e^{-ct}h(z)\right),$$

where $h : \mathbb{D} \to h(\mathbb{D}) = \Omega$ is a univalent function with $\Omega$ a spirallike domain, $h(0) = 0$, $Re\, c \geq 0$ and $\omega e^{-ct} \in \Omega$ for each $\omega \in \Omega$, $t \geq 0$.

## Denjoy-Wolff point in the boundary

If $b \in \partial\mathbb{D}$ it may be reduced to $b = 1$. Then

$$\varphi_t(z) = h^{-1}\left(h(z) + ct\right),$$

where $h : \mathbb{D} \to h(\mathbb{D}) = \Omega$ is a univalent function with $\Omega$ a close-to-convex domain, $h(0) = 0$, $Re\, c \geq 0$ and $\omega + ct \in \Omega$ for each $\omega \in \Omega$, $t \geq 0$.

This connection between composition operators $(C_t)$ and semigroups $(\varphi_t)$ opens the possibility of studying properties of the semigroup of operators $(C_t)$ in terms of the theory of functions.

## Some results

- Every semigroup $(\varphi_t)$ generates a semigroup of operators on the Hardy spaces $H^p$ $(1 \leq p < \infty)$, the Bergman spaces $A^p$ $(1 \leq p < \infty)$, the Dirichlet space, and on the spaces *VMOA* and little Bloch.
- No non-trivial semigroup generates a semigroup of operators in spaces $H^\infty$, *BMOA*, $\mathcal{B}$ or $\mathcal{L}^{2,\lambda}, 0 < \lambda < 1$.
- There are plenty of semigroups (but not all) which generate semigroups of operators in the disc algebra $\mathcal{A}$.

**Theorem**

Let be $0 < \lambda < 1$ and $(\varphi_t)$ a semigroup of analytic functions. Then there exists a closed subspace $Y \subset \mathcal{L}^{2,\lambda}$ such that $(\varphi_t)$ generates a semigroup of operators on $Y$ and such that any other subspace of $\mathcal{L}^{2,\lambda}$ with this property is contained in $Y$.

We write that space $Y$ as $[\varphi_t, \mathcal{L}^{2,\lambda}]$.

## Theorem

Let be $0 < \lambda < 1$ and $(\varphi_t)$ a semigroup of analytic functions. Let $G$ the infinitesimal generator of $(\varphi_t)$ then

$$[\varphi_t, \mathcal{L}^{2,\lambda}] = \overline{\{f \in \mathcal{L}^{2,\lambda} : Gf' \in \mathcal{L}^{2,\lambda}\}}.$$

## Theorem

For $0 < \lambda < 1$, every semigroup $(\varphi_t)$ generates a semigroup of operators on $\mathcal{L}_0^{2,\lambda}$.

$$\mathcal{L}_0^{2,\lambda} \subseteq [\varphi_t, \mathcal{L}^{2,\lambda}] \subseteq \mathcal{L}^{2,\lambda} \quad 0 < \lambda < 1.$$

The inclusion $\mathcal{L}_0^{2,\lambda} \subseteq [\varphi_t, \mathcal{L}^{2,\lambda}]$ can be proper.
For the semigroup $\varphi_t(z) = e^{-t}z + 1 - e^{-t}, \quad t \geq 0, z \in \mathbb{D}$ the function $f(z) = (1-z)^{-\frac{1-\lambda}{2}} \in \mathcal{L}^{2,\lambda} \setminus \mathcal{L}_0^{2,\lambda}$ satisfies

$$\begin{aligned} \|f \circ \varphi_t - f\|_{\mathcal{L}^{2,\lambda}} &= \|e^{t\frac{1-\lambda}{2}}(1-z)^{\frac{1-\lambda}{2}} - (1-z)^{\frac{1-\lambda}{2}}\|_{\mathcal{L}^{2,\lambda}} \\ &= C\left(e^{t\frac{1-\lambda}{2}} - 1\right) \to 0. \end{aligned}$$

At this point it is natural to ask about conditions in $(\varphi_t)$ such that $\mathcal{L}^{2,\lambda}_0 = [\varphi_t, \mathcal{L}^{2,\lambda}]$ or $\mathcal{L}^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

## Conditions for $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$

### Theorem

Let $(\varphi_t)$ be a semigroup with infinitesimal generator $G$ and $0 < \lambda < 1$. Assume that for some $0 < \alpha < 1/2$,

$$\frac{(1-|z|)^\alpha}{G(z)} = O(1) \quad \text{as } |z| \to 1.$$

Then $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

We can prove that as a consequence of a stronger theorem.

**Theorem**

Let $(\varphi_t)$ be a semigroup with infinitesimal generator $G$ and $0 < \lambda < 1$. Assume that

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{S(I)} \frac{1 - |z|}{|G(z)|^2} \, dA(z) = 0.$$

Then $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$.

## Theorem

Let $(\varphi_t)$ be a semigroup with infinitesimal generator $G$ and Denjoy-Wolff point $b \in \mathbb{D}$.
If $\mathcal{L}_0^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$ then

$$\lim_{|z|\to 1} \frac{(1-|z|)^{\frac{3-\lambda}{2}}}{G(z)} = 0.$$

## Problem

Are there semigroups such that $\mathcal{L}^{2,\lambda} = [\varphi_t, \mathcal{L}^{2,\lambda}]$ ?

## Theorem (BCD-MMPS)

There are no non-trivial semigroups such that $[\varphi_t, \mathcal{B}] = \mathcal{B}$.

Proof: It is strongly used that every $\mathcal{B}^\alpha$ is a Grothendieck space with the Dunford-Pettis property.

We do not know if Morrey spaces $\mathcal{L}^{2,\lambda}$, $0 < \lambda < 1$ satisfy this property. *BMOA* does not have it.

This question has remained open for *BMOA* until 2017.

**Theorem (Anderson, Jovovic, Smith. 2017)**

Suppose $H^\infty \subset X \subset \mathcal{B}$. Then there are no non-trivial semigroups such that $[\varphi_t, X] = X$.

**Theorem**

Suppose $0 < \lambda < 1$ and $\mathcal{L}^{2,\lambda} \subset X \subset \mathcal{B}^{\frac{3-\lambda}{2}}$. Then there are no non-trivial semigroups such that $[\varphi_t, X] = X$.

**Corollary**

For $0 < \lambda \leq 1$ there are no non-trivial semigroups such that $[\varphi_t, \mathcal{L}^{2,\lambda}] = \mathcal{L}^{2,\lambda}$.

## Proof

Given any non-trivial semigroup $(\varphi_t)$ and $0 < \lambda < 1$ we just need a function $f \in \mathcal{L}^{2,\lambda}$ such that

$$1 \leq \liminf_{t \to 0} \|f \circ \varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} .$$

If the Denjoy-Wolff point of $(\varphi_t)$ is $b = 0$ then

$$\varphi_t(z) = h^{-1}\left(e^{-ct} h(z)\right) .$$

When $Re\, c = 0$ the $(\varphi_t)$ are rotations of the disc.

## Proof

The function $f(z) = (1-z)^{-\frac{1-\lambda}{2}} \in \mathcal{L}^{2,\lambda}$ and

$$\lim_{r\to 1^-} |f'(r)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2} > 0.$$

$$\lim_{r\to 1^-} |f'(re^{i\theta})|(1-r)^{\frac{3-\lambda}{2}} = 0 \quad \text{for } \theta \neq 0.$$

So if $\varphi_t(z) = ze^{iat}$ for real $a \neq 0$ for $t \in \left(0, \frac{2\pi}{|a|}\right)$

$$\|f\circ\varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} \geq \sup_{0<r<1} |f'(\varphi_t(r))\varphi_t'(r) - f'(r)|(1-r)^{\frac{3-\lambda}{2}} \geq \frac{1-\lambda}{2}.$$

## Proof

If $Re\, c > 0$, $(\varphi_t)$ does not consist of automorphisms.
Since $\Omega$ is spirallike about 0, we can choose $\omega_0 \in \partial\Omega$ such that

$$|\omega_0| = \inf\{|\omega| : \omega \in \partial\Omega\}.$$

Since $h$ is univalent there is a $\gamma_0 \in \partial\mathbb{D}$ such that $\lim_{r\to 1^-} h(r\gamma_0)$ exists and is equal to $\omega_0$. Thus,

$$\lim_{r\to 1^-} \varphi_t(r\gamma_0) = h^{-1}(e^{-ct}\omega_0) \in \mathbb{D}, \quad t > 0.$$

Since $\varphi_t \in \mathcal{U} \cap H^\infty \subset \mathcal{D} \subset \mathcal{B}_0^{\frac{3-\lambda}{2}}$ $t \geq 0$ we have

$$\lim_{r\to 1^-} |\varphi_t'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} = 0.$$

## Proof

Letting $f(z) = (1 - \overline{\gamma_0} z)^{-(1-\lambda)/2}$, we have

$$\lim_{r \to 1^-} |f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} = \frac{1-\lambda}{2} > 0.$$

Thus for all $t \geq 0$

$$\begin{aligned} \|f \circ \varphi_t - f\|_{\mathcal{B}^{\frac{3-\lambda}{2}}} &\geq \limsup_{r \to 1^-} |f'(\varphi_t(r\gamma_0))\varphi_t'(r\gamma_0) - f'(r\gamma_0)|(1-r)^{\frac{3-\lambda}{2}} \\ &\geq \frac{1-\lambda}{2}. \end{aligned}$$

If $b = 1$ we use also $f(z) = (1 - \overline{\gamma_0} z)^{-(1-\lambda)/2}$.

# THANK YOU!

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