On the solutions of the incompressible Euler equations

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Harmonic Maps and Ideal Fluid Flows

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Incompressible Euler Equations: Eulerian frame

Incompressible inviscid flows are described by the equation of mass conservation

$$u_x + v_y = 0$$

coupled with the Euler equations

$$\begin{cases} u_t + uu_x + vu_y = -P_x \\ v_t + uv_x + vv_y = -P_y \end{cases},$$

where $(u(t,x,y), v(t,x,y))$ is the velocity field in the time and space variables (t,x,y) and the scalar function $P(t,x,y)$ represents the pressure.

For a given velocity field $(u(t,x,y), v(t,x,y))$, the motion of the individual particles $(x(t), y(t))$ is obtained by integrating the system of ordinary differential equations

$$\left\{ \begin{array}{l} x'(t) = u(t,x,y) \\ y'(t) = v(t,x,y) \end{array} \right.$$

whereas the knowledge of the particle path $t \mapsto (x(t), y(t))$ provides by differentiation with respect to t the velocity field at time t and at the location $(x(t), y(t))$.

Incompressible Euler Equations: Lagrangian coordinates

Starting with a simply connected domain Ω_0, representing the labelling domain, each label $(a, b) \in \Omega_0$ identifies by means of the injective map

$$(a, b) \mapsto F^t(a, b) = (x(t, a, b), y(t, a, b))$$

the evolution in time of a specific particle, the fluid domain at time t, Ω_t, being the image of Ω_0 under F^t.

The governing equations in Lagrangian coordinates

Using the relations

$$\begin{cases} \frac{\partial}{\partial a} & = & x_a \frac{\partial}{\partial x} + y_a \frac{\partial}{\partial y} \\ \frac{\partial}{\partial b} & = & x_b \frac{\partial}{\partial x} + y_b \frac{\partial}{\partial y} \end{cases},$$

we see that the equation of mass conservation becomes

$$J_t = 0\,.$$

Euler's equations take the form

$$(x_a x_{bt} + y_a y_{bt} - x_b x_{at} - y_b y_{at})_t = 0\,.$$

Explicit solutions

- Gerstner's flow (found in 1809 and re-discovered in 1863 by Rankine):

$$\begin{aligned} F^t(a,b) &= (x(t,a,b), y(t,a,b)) \\ &= \left(a + \frac{e^{kb}}{k}\sin(k(a+ct)), b - \frac{e^{kb}}{k}\cos(k(a+ct))\right), \end{aligned}$$

  where $kc^2 = g$ and $(a,b) \in \Omega_0 = \{(a,b) \colon b < 0\}$,
- Kirchhoff's elliptical vortex, found in 1876,
- and the Ptolemaic vortices found in 1984 by Abrashkin and Yakubovich.

Gerstners flow (1809)

$$\begin{aligned} F^t(a,b) &= (x(t,a,b), y(t,a,b)) \\ &= \left(a + \frac{e^{kb}}{k}\sin(k(a+ct)), b - \frac{e^{kb}}{k}\cos(k(a+ct))\right) . \end{aligned}$$

Use

$$(a,b) \approx a + ib = z \quad \text{and} \quad (x,y) \approx x + iy = F = f + \overline{g}\,,$$

where $z \in \Omega_0 = \{z \in \mathbb{C} \colon Im\{z\} < 0\}$ and f and g are analytic in Ω_0 because

$$x_{aa} + x_{bb} = y_{aa} + y_{bb} = 0!$$

A. Aleman and A. Constantin: find all solutions which in Lagrangian variables present a labelling by harmonic functions.

Theorem: Assume that there exist $z_1, z_2 \in \Omega_0$ and an open set $I \subset (0, \infty)$ such that for all $t \in I$ the vectors

$$\begin{pmatrix} f'(t, z_j) \\ i g_t(t, z_j) \end{pmatrix}_{j=1,2}$$

are linearly independent. The solutions $f(t, z) + \overline{g(t, z)}$ are then given by

$$\begin{pmatrix} f(t, z) \\ g(t, z) \end{pmatrix} = \begin{pmatrix} \alpha(t) & \beta(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} u_0(z) \\ v_0(z) \end{pmatrix},$$

where u_0' and v_0' are linearly independent, $\alpha d - \beta c \neq 0$ on I, and

$$\begin{cases} A'\overline{A} - c\overline{c'} & = & ik_1 \\ B'\overline{B} - d\overline{d'} & = & ik_2 \\ B'\overline{A} - d\overline{c'} & = & k_3 \\ A'\overline{B} - c\overline{d'} & = & -\overline{k_3}\,, \end{cases}$$

where $k_1, k_2 \in \mathbb{R}$, $k_3 \in \mathbb{C}$, $A' = \alpha$, and $B' = \beta$.

- Let Ω_0 be a convex domain whose boundary does not contain line segments, and let f, g be analytic functions in Ω_0 whose derivatives extend continuously to $\overline{\Omega_0}$ and satisfy

$$Re\{f'(z)\} > |g'(z)|\,, \quad z \in \Omega_0\,.$$

Then the harmonic map $z \mapsto f(z) + \overline{g(z)}$ is univalent (one-to-one) in Ω_0.

Harmonic mappings

A harmonic mapping F in a simply connected domain $\Omega \subset \mathbb{C}$ can be written as

$$F = f + \overline{g}\,,$$

where both f and g are analytic in Ω.

- F is analytic if and only if g is constant,
- If F is harmonic and φ is analytic, then $F \circ \varphi$ is harmonic,
- Given a harmonic mapping F, the composition $A \circ F$, where A is an affine harmonic mapping of the form

  $$A(z) = az + b\overline{z} + c\,, \quad a, b, c \in \mathbb{C}\,,$$

  is harmonic as well.

Harmonic mappings

$$F = f + \overline{g}\,.$$

Lewy (1936) F is *locally univalent* if and only if its Jacobian

$$J_F = |f'|^2 - |g'|^2 = |f'|^2(1 - |\omega|^2) \neq 0\,.$$

Here, $\omega = \overline{F_{\overline{z}}}/F_z = g'/h'$ is the (second complex) dilatation of F.

A locally univalent harmonic mapping is orientation-preserving if $J_F > 0$ (that is, if -and only if- f is locally univalent and ω is an analytic function with $\|\omega\|_\infty \leq 1$).

The harmonic Koebe function

$$K = f + \overline{g}\,,$$

where f and g are the analytic functions in $\mathbb{D}$ given by

$$f(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} = z + \frac{5}{2}z^2 + \sum_{n=3}^{\infty} a_n z^n$$

and

$$g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} = \sum_{n=2}^{\infty} b_n z^n\,.$$

- K is univalent (one-to-one) in $\mathbb{D}$ and satisfies $f(0) = g(0) = 1 - f'(0) = 0$, $g'(0) = 0$. Also,

$$\begin{cases} f(z) - g(z) = k(z) = \frac{z}{(1-z)^2} \\ g'(z)/f'(z) = z \end{cases}, \quad z \in \mathbb{D}\,.$$

Univalent harmonic mappings in the unit disk

A harmonic mapping $F = f + \overline{g}$ in the unit disk belongs to the class $\mathcal{S}_H$ if it is orientation-preserving, univalent in $\mathbb{D}$, and satisfies

$$f(0) = g(0) = 1 - f'(0) = 0.$$

The functions

$$F_n(z) = z + \frac{n}{n+1}\overline{z} \in \mathcal{S}_H\,.$$

$$\mathcal{S}_H^0 = \{F \in \mathcal{S}_H \colon g'(0) = 0\}\,.$$

- If $F \in \mathcal{S}_H$, then

$$\frac{F - \overline{\omega(0)F}}{1 - |\omega(0)|^2} = A \circ F \in \mathcal{S}_H^0\,.$$

The Schwarzian derivative

The Schwarzian derivative of a locally univalent analytic function φ in the unit disk is defined by

$$S(\varphi) = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 = (P(\varphi))' - \frac{1}{2}(P(\varphi))^2 \ ,$$

where $P(\varphi)$ is the pre-Schwarzian derivative of φ.

Remark.

$$P(\varphi) = \frac{\varphi''}{\varphi'} = \frac{\partial}{\partial z}\left(\log|\varphi'|^2\right) = \frac{\partial}{\partial z}\left(\log J_\varphi\right) \ .$$

Therefore,

$$S(\varphi) = \frac{\partial^2}{\partial z^2}\left(\log J_\varphi\right) - \frac{1}{2}\left(\frac{\partial}{\partial z}\left(\log J_\varphi\right)\right)^2 \ .$$

The Schwarzian derivative (& Newton & Halley)

Discovered by Lagrange in his treatise "*Sur la construction des cartes gographiques*" (1781); the Schwarzian also appeared in a paper by Kummer (1836), and it was named after Schwarz by Cayley. However, this operator comes up naturally in the numerical method of approximation of zeros of functions due to Halley (1656-1742)!

The Schwarzian derivative (& Newton & Halley)

Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \equiv \quad \alpha \approx x - \frac{f(x)}{f'(x)}$.

$$0 = f(\alpha) \approx f(x) + f'(x)(\alpha - x) + \frac{f''(x)}{2}(\alpha - x)^2 + \dots$$

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$$0 = f(\alpha) \approx f(x) + f'(x)(\alpha - x) + \cancel{\frac{f''(x)}{2}(\alpha - x)^2} + \cancel{\dots}$$

$$\alpha \approx x - \frac{f(x)}{f'(x)} = F_N(x)\,.$$

A straightforward calculation shows $F_N''(\alpha) = \frac{f''(\alpha)}{f'(\alpha)} = P(f)(\alpha)$.

$$\alpha - x \approx -\frac{f(x)}{f'(x)}$$

The Schwarzian derivative (& Newton & Halley)

Halley's method: $x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}$.

$$\equiv \alpha \approx x - \frac{2f(x)f'(x)}{2\left(f'(x)\right)^2 - f(x)f''(x)} = F_H(x)\,.$$

$$\begin{aligned} 0 = f(\alpha) &\approx f(x) + f'(x)(\alpha - x) + \frac{f''(x)}{2}(\alpha - x)^2 + \cancel{\ldots} \\ &\approx f(x) + (\alpha - x)\left(f'(x) + \frac{f''(x)}{2}(\alpha - x)\right) \\ &\approx f(x) + (\alpha - x)\left(f'(x) + \frac{f''(x)}{2}\left(-\frac{f(x)}{f'(x)}\right)\right)\,. \end{aligned}$$

And... $F_H'''(\alpha) = -S(f)(\alpha)$.

The Schwarzian derivative

$$S(\varphi) = \left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 .$$

- If the composition $\varphi \circ \psi$ is well defined,

$$S(\varphi \circ \psi) = S(\varphi)(\psi) \cdot (\psi')^2 + S(\psi) .$$

- The Schwarzian norm or the locally univalent function φ in $\mathbb{D}$ equals

$$\|S(\varphi)\| = \sup_{z \in \mathbb{D}} |S(\varphi)(z)|(1 - |z|^2)^2 .$$

Univalence criteria

Let φ be a locally univalent analytic function in $\mathbb{D}$.

- (Becker, 1962) If

$$\|P(\varphi)\| = \sup_{z\in\mathbb{D}} |P(\varphi)(z)|(1-|z|^2) \leq 1$$

or

- (Nehari, 1949) If

$$\|S(\varphi)\| = \sup_{z\in\mathbb{D}} |S(\varphi)(z)|(1-|z|^2)^2 \leq 2\,,$$

then φ is globally univalent in $\mathbb{D}$.

The harmonic Schwarzian derivative

The harmonic Schwarzian derivative of the locally univalent harmonic mapping F is defined by

$$\begin{aligned} S_H(F) &= \frac{\partial^2}{\partial z^2}(\log J_F) - \frac{1}{2}\left(\frac{\partial}{\partial z}(\log J_F)\right)^2 \\ &= \frac{\partial}{\partial z}(P_H(F)) - \frac{1}{2}(P_H(F))^2 \, . \end{aligned}$$

- $S_H(F) = S_H(\overline{F})$ and $S_H(f+\overline{g}) = S_H(f+\overline{\mu g})$ for all $|\mu| = 1$.
- If $F = f + \overline{g}$ is an orientation preserving harmonic mapping with dilatation $\omega = g'/f'$,

$$S_H(F) = S(f) - \frac{\overline{\omega}}{1-|\omega|^2}\left(\omega' \frac{f''}{f'} - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2 .$$

The harmonic Schwarzian derivative

$$S_H(F) = S(f) - \frac{\overline{\omega}}{1-|\omega|^2}\left(\omega' \frac{f''}{f'} - \omega''\right) - \frac{3}{2}\left(\frac{\overline{\omega}\,\omega'}{1-|\omega|^2}\right)^2 .$$

- If F is analytic then $S_H(F) = S(F)$.
- Let F be orientation-preserving harmonic mapping and let φ be an analytic function such that the composition $F \circ \varphi$ is well-defined. Then

$$S_H(F \circ \varphi) = S_H(F)(\varphi) \cdot \left(\varphi'\right)^2 + S(\varphi).$$

- Let A be a locally univalent affine harmonic mapping. That is, $A(z) = az + b\overline{z} + d$, where $|a| \neq |b|$. Then

$$S_H(A \circ F) = S_H(F)\,.$$

Univalence criteria

Let F be a locally univalent harmonic function in $\mathbb{D}$.

- If

$$\sup_{z\in\mathbb{D}}\left(|P_H(F)(z)|(1-|z|^2)+\frac{|\omega'(z)|(1-|z|^2)}{1-|\omega(z)|^2}\right)\leq 1$$

  or

-

$$\|S_H(F)\|=\sup_{z\in\mathbb{D}}|S_H(F)(z)|(1-|z|^2)^2\leq\delta_0\,,$$

then F is globally univalent in $\mathbb{D}$.

Moreover,

Two locally univalent functions F_1 and F_2 on a simply connected domain Ω_0 with non-constant dilatation have equal harmonic pre-Schwarzian derivative if and only if there exists an affine transformation A and an anti-analytic rotation R_μ such that

$$F_2 = (A \circ R_\mu)(F_1)\,.$$

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Wait...

$$P_H(F) = \frac{\partial}{\partial z}\left(\log J_F\right)\,.$$

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$$P_H(F) = \frac{\partial}{\partial z}\left(\log J_F\right)\,.$$

And we obtain the following relation between two harmonic functions $F_1 = f_1 + \overline{g_1}$ and $F_2 = f_2 + \overline{g_2}$ with equal Jacobian:

$$\begin{pmatrix} f_2 \\ g_2 \end{pmatrix} = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix},$$

where $|a|^2 - |b|^2 = 1$.

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where $|a|^2 - |b|^2 = 1$. Wait...the mass conservation equation reads... $J_t = 0$!!

Theorem

Let $\Omega_0 \subset \mathbb{C}$ be a simply connected domain. Assume that the initial harmonic (univalent, orientation-preserving) labelling map $F_0 = f_0 + \overline{g_0}$ is such that f_0' and g_0' are linearly independent. The particle motion of a fluid flow in Lagrangian coordinates is either described by

$$\begin{pmatrix} f(t,z) \\ g(t,z) \end{pmatrix} = \begin{pmatrix} \frac{a(t)}{b(t)} & \frac{b(t)}{a(t)} \end{pmatrix} \begin{pmatrix} f_0(z) \\ g_0(z) \end{pmatrix},$$

where $b : [0, \infty) \to \mathbb{C}$ is a C^1 function and

$$a(t) = \sqrt{1 + |b(t)|^2}\, e^{i \int_0^t \frac{\nu_0 + Im\{b_t(s)\overline{b(s)}\}}{1+|b(s)|^2} ds}, \quad \nu_0 \in \mathbb{R},$$

or

$$\begin{pmatrix} f(t,z) \\ g(t,z) \end{pmatrix} = \begin{pmatrix} e^{i\nu_0 t} & 0 \\ 0 & e^{i(\nu_0-\xi_0)t} \end{pmatrix} \begin{pmatrix} f_0(z) \\ g_0(z) \end{pmatrix},$$

where $\nu_0 \in \mathbb{R}$ and $\xi_0 \in \mathbb{R} \setminus \{0\}$.

Univalence holds for the solutions in this second case for all the functions F^t if and only if $f_0 + \lambda \overline{g_0}$ is univalent for all $|\lambda| = 1$.

The constant dilatation case

If the initial harmonic labelling map $F_0 = f_0 + \overline{g_0}$ satisfies $g_0' = cf_0'$ for some $|c| < 1$, then

$$\begin{pmatrix} f(t,z) \\ g(t,z) \end{pmatrix} = \begin{pmatrix} a(t) & 0 \\ 0 & b(t) \end{pmatrix} \begin{pmatrix} f_0(z) \\ g_0(z) \end{pmatrix},$$

where $b : [0, \infty) \to \mathbb{C}$ is a C^1 function with $b(0) = c$ and

$$a(t) = \sqrt{1 - |c|^2 + |b(t)|^2}\, e^{i \int_0^t \frac{\nu_0 + Im\{b_t(s)\overline{b(s)}\}}{1-|c|^2+|b(s)|^2}\, ds}, \quad \nu_0 \in \mathbb{R}\,.$$

Thank you very much for your attention!