On the branch set of mappings of finite and bounded distortion.

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Branched covers, quasiregular mappings and MFD

Holomorphic mappings are always *continuous*, *open* and *discrete*. By the classical *Stoïlow theorem*, the converse also holds; a continuous open and discrete map in the plane is holomorphic up to a homeomorphic reparametrization.
In higher dimensions one of the classical generalizations of holomorphic mappings is the class of *quasiregular maps*:

Definition

A mapping $f\colon \Omega \to \mathbb{R}^n$ is K-quasiregular if $f \in W^{1,n}$ and

$$\|Df(x)\|^n \leq K J_f(x)$$

for almost every $x \in \Omega$.

By Reshetnyak's theorem, quasiregular mappings are always continuous, open and discrete.

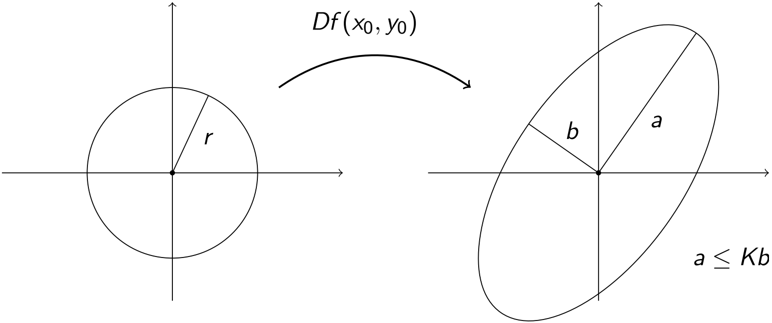


Figure: The canonical picture describing quasiregular mappings via the behaviour of their tangent maps.

We call a continuous, open and discrete mapping a *branched cover*. The set of points where a branched cover f fails to be a local homeomorphism is called *the branch set* of the mapping and we denote it by B_f.
For planar mappings the branch set is a discrete set (think $z \mapsto z^2$). More generally for branched covers between euclidean n-domains the branch set has *topological dimension* of at most $n - 2$.

What can the branch set look like in general?

- Can the branch set of a branched cover $\mathbb{R}^3 \to \mathbb{R}^3$ be a Cantor set? (Church-Hemmingsen 1960)
- Can the branch set of a proper branched cover $B^n(0, 1) \to \mathbb{R}^n$ be compact? (Vuorinen 1979)
- Can we describe the geometry and the topology of branch set of quasiregular mappings? (Heinonen's ICM address 2002)

More non-trivial examples are needed in order to understand this problem.

Theorem

For every $n \geq 3$ there exists a branched cover $\mathbb{R}^n \to \mathbb{R}^n$ with the branch set equal to the $(n-2)$-dimensional torus.

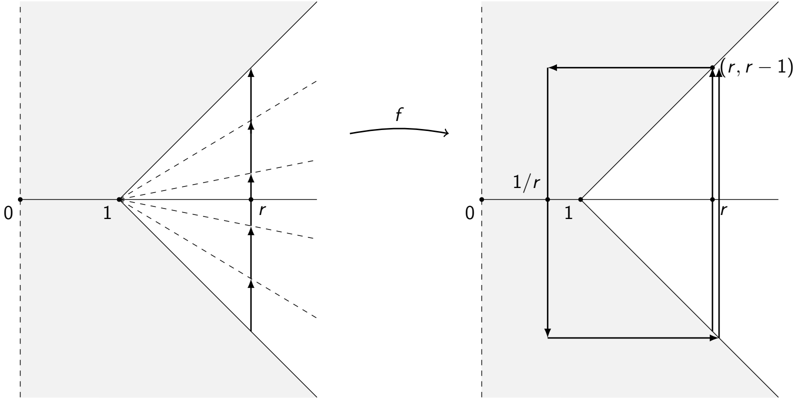
Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular mapping, $n \geq 3$. Then the branch set is either empty or unbounded.

Constructing the example map F in three dimensions

By T_α we denote for each $\alpha \in [0, 2\pi)$ the half plane forming angle α with the plane $T_0 = \{(x, 0, z) : x \geq 0\}$.
The mapping $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ will map each half-plane T_α onto itself and the restrictions $F|_{T_\alpha}$ will be topologically equivalent to the complex winding map $z \mapsto z^2$.
We define our mapping on each of the closed half-planes $\overline{T_\alpha}$. The restrictions will be similar and we denote any and all of the restrictions as f.

On each half-plane the mapping equals a so-called sector winding:



Since the branch of each of these half-plane mappings has a singleton branch set, we see that $B_F = \mathbb{S}^1 \times \{0\}$.

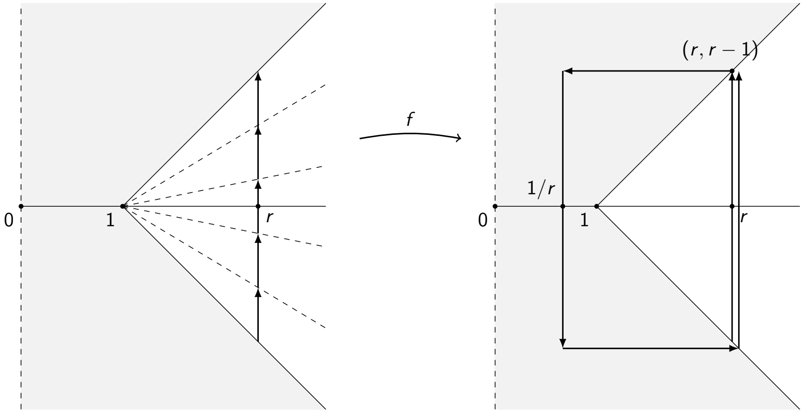
Proof of the positive statement

Suppose f is a quasiregular mapping with branch set contained in the open unit ball.

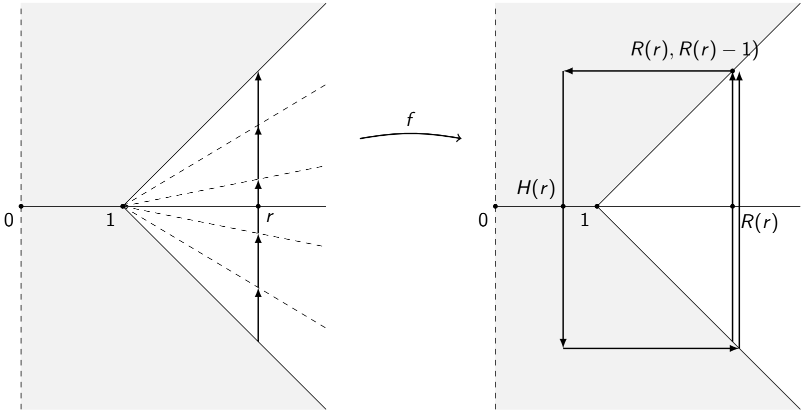
- Take $h\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ to be the conformal reflection with respect to the sphere.
- Set $g := (f|_{\mathbb{R}^n \setminus \overline{B}(0,1)}) \circ h\colon B(0,1) \setminus \{0\} \to \mathbb{R}^n$
- The mapping g is now a locally homeomorphic quasiregular mapping.
- By a result of Agard and Marden (1971) such a mapping extends to a local homeomorphism to the whole ball if and only if a certain modulus condition holds for the image of the collection of paths touching the origin. $(M(g(\Gamma_0)) = 0)$
- The condition is translates to asking if $M(f(\Gamma_\infty)) = 0$.
- It happens to hold for quasiregular mappings!
- Thus the original mapping f extends to $\hat{f}\colon \mathbb{S}^n \to \mathbb{S}^n$
- By topological degree theory, this implies that the infinity point is an isolated branch point, which is impossible in dimensions 3 and above by classical results of Church and Hemmingsen (1960).

What is the extent of these results?

- How badly not-quasiregular is the example map?
- For which class of branched covers does $M(f(\Gamma_\infty)) = 0$ hold?.



0
1
r
f
(r, r − 1)
1/r
0
1
r


0
1
r
f
H(r)
0
1
R(r), R(r) − 1)
R(r)

Actual form of main theorems

Definition

A mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, defined on an open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, is called a *mapping of finite distortion* if $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and

$$\|Df(x)\|^n \leq K_f(x) J_f(x)$$

for almost every $x \in \Omega$ where $K_f \in L^1_{\mathrm{loc}}$.

Mappings of finite distortion are also branched covers under some mild integrability conditions for K_f.

Actual form of main theorems

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping of finite distortion, $n \geq 3$. Suppose that f is a branched cover and

$$K_f(x) \leq o(\log(\|x\|))$$

away from origin. Then the branch set is either empty or unbounded.

Theorem

For every $n \geq 3$ and every $\varepsilon > 0$ there exists a piecewise smooth branched cover $\mathbb{R}^n \to \mathbb{R}^n$ such that f has a branch set equal to the $(n-2)$-dimensional torus and $K_f(x) \leq (\log(\|x\|))^{1+\varepsilon}$.

Final remarks

- We don't know what happens when $K_f \sim \log(\|x\|)$.
- The example does not answer the question of Vuorinen.
- This is yet another mapping that is essentially a clever winding map.
- More examples of compact branch sets can be extracted from the example.

Euχαρíστω!