# Finite Rank Perturbations

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# Setting and definitions

Given operator $A$, what can we say about the spectral properties of

$$A + B \quad \text{for} \quad B \in \text{Class } X?$$

- Classically Class $X = \{\text{trace cl.}\}, \{\text{Hilb.–Schmidt}\}, \{\text{comp.}\}$.
- Here: $A$, $B$ self-adj. on separable $\mathcal{H}$, Class $X = \{\text{finite rk}\}$.

### Definition

Through $A_\gamma = A + \gamma(\,\cdot\,, \varphi)\varphi$, parameter $\gamma \in \mathbb{R}$ realizes all *self-adjoint rank one perturbations* (of a given self-adjoint operator $A$) in the direction of a cyclic $\varphi$ (WLOG).

### Definition

Through $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$, the symmetric $d \times d$ matrices $\Gamma$ parametrize all *self-adjoint finite rank perturbations* with range contained in that of $\mathbf{B}$. WLOG: Range $\mathbf{B}$ is a cyclic subspace and $\mathbf{B} : \mathbb{C}^d \to \mathcal{H}$ left-invertible on its range.

# Classical perturbation theory ($A$, $T = A + B$ self-adjoint)

- Notation $A \sim T$ means that $UA = TU$ with unitary $U$, and
- $A \sim T$(Mod compact operators) means $UA = TU + K$ for some unitary $U$ and compact $K$.

**Theorem (Weyl–vonNeuman early 1900's)**

$A \sim T$(*Mod compact operators*) $\Leftrightarrow$ $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(T)$.

**Theorem (Kato–Rosenblum 1950's, Carey–Pincus 1976)**

$A \sim T$(*Mod trace class*) $\Leftrightarrow$ $A_{\mathrm{ac}} \sim T_{\mathrm{ac}}$, *conditions.*

**Theorem (Aronszajn–Donoghue Theory 1970-80's)**

*Spectral type is not stable under rank one perturbations. Complete information about the eigenvalues, but only a set outside which* $A_\gamma$ *has no singular continuous spectrum. (see later)*

**Theorem (Poltoratski 2000)**

*Conditions on purely singular operators* $\Rightarrow A \sim T$(*Mod rank 1*).

# Subset of interested people

Unitary rank one perturbations or their corresponding model spaces were studied by Aleksandrov, Ball, Clark, Douglas–Shapiro–Shields, Kapustin, Poltoratski, Ross, Sarasson, etc.

A self-adjoint setting was studied by Albeverio–Kurasov, Aronszajn–Donoghue, delRio, Kato-Rosenblum, Simon, etc.

Finite rank generalizations occur in literature by Albeverio–Kurasov (extension theory), Gesztesy et al., Kapustin–Poltoratski (no a.c.), Martin.

# What are rank one perturbations related to?

## In mathematical physics

- Half-line Schrödinger operator $Hu = -\frac{d^2}{dx^2}u + Vu$ with changing boundary condition (Weyl 1910)
- Anderson-type Hamiltonian $H_\omega = H + \sum_{m=1}^{\infty} \omega_m(\,\cdot\,, \varphi_m)\varphi_m$ for orthonormal $\varphi_m$ and i.i.d. random $\omega_m$ wrt $\mathbb{P}$

## Within analysis

- Extension theory of symmetric operators:
  - Changing boundary conditions of Sturm–Liouville operators
  - Changing boundary conditions for PDEs
- Nehari interpolation problem
- Holomorphic composition operators
- Rigid functions
- Functional models (Nagy–Foiaș, deBr.–Rovn., Nik.–Vasyunin)
- Two weight problem for Hilbert/Cauchy transform
- Carlesson embedding

# What are finite rank perturbations related to?

- Describe all self-adjoint extensions of a symmetric operator with finite deficiency indicees $(d, d)$
- Functional models with matrix-valued characterisic functions (Nagy–Foiaș, deBr.–Rovn., Nik.–Vasyunin)
- Two weight problem for Hilbert/Cauchy transform with matrix-valued weights

# Finite dimensional examples (recall $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$)

- $A_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma(\,\cdot\,, e_1)e_1 = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 3 \end{pmatrix}$ acting on $\mathbb{R}^2$.
  Here $e_1$ is not cyclic.
- $A_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma(\,\cdot\,, e_1+e_2)(e_1+e_2) = \begin{pmatrix} 1+\gamma & \gamma \\ \gamma & 3+\gamma \end{pmatrix}$.
  Here $e_1 + e_2$ is cyclic.
- $A_{\gamma_1,\gamma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma_1(\,\cdot\,, e_1)e_1 + \gamma_2(\,\cdot\,, e_2)e_2 = \begin{pmatrix} 1+\gamma_1 & 0 \\ 0 & 3+\gamma_2 \end{pmatrix}$.
  Even if $\gamma_1 = \gamma_2$, this cannot be written as rank one perturbation; The $\{e_1, e_2\}$ spans a cyclic subspace.
- Cyclicity of $A$ does not necessarily imply that of $A_\Gamma$:
  For $\gamma_1 = \gamma_2 - 2$, $A_{\gamma_1,\gamma_2} = \begin{pmatrix} 1+\gamma_1 & 0 \\ 0 & 3+\gamma_2 \end{pmatrix}$ has one mult. 2 EVA. Otherwise, there are two EVA each of mult. 1.

For a $k \times k$ matrix, the $k$ eigenvalues depend on the parameters. Finding EVA and EVE consists of diagonalization $UA_\gamma = DU$. Operators on infinite dimensional space (e.g. Hilbert space) reveal more complicated spectral behavior.

# Scalar measure and decomposition

**Theorem (Scalar Spectral Theorem)**

*Let $A$ be a self-adjoint operator on Hilbert space $\mathcal{H}$ with (cyclic) vector $\varphi$. Then there exists a unique measure $\mu = \mu^{\varphi}$ such that*

$$\left((A-\lambda\mathbf{I})^{-1}\varphi,\varphi\right)_{\mathcal{H}} = \int_{\mathbb{R}} \frac{d\mu(t)}{t-\lambda} = \left((M_t-\lambda\mathbf{I})^{-1}\mathbf{1},\mathbf{1}\right)_{L^2(\mu)}$$

*for $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Namely, $A \sim M_t$ on $L^2(\mu)$.*

- $\mu$ contains all the spectral information of operator $A$.
- EVA $\lambda$ of $A$ is reflected in point mass at $\lambda$, i.e. $\mu\{\lambda\} > 0$.
- Lebesgue decompose the spectral measure $d\mu = d\mu_{\mathrm{ac}} + d\mu_{\mathrm{s}}$.
- Further decompose $d\mu_{\mathrm{s}} = d\mu_{\mathrm{p}} + d\mu_{\mathrm{sc}}$.
- Through $A \sim M_t$ decompose operator $A = A_{\mathrm{ac}} \oplus A_{\mathrm{p}} \oplus A_{\mathrm{sc}}$.

# Matrix-valued spectral measures

Define $b_k := \mathbf{B}\mathbf{e}_k$, for $k = 1, 2, \ldots, d$. Consider *(singular) form bounded perturbations*, that means that for each $k$ we have $\|(1 + |A|)^{-1/2} b_k\|_{\mathcal{H}} < \infty$ where $|A| = (A^*A)^{1/2}$.

**Theorem (Matrix-valued Spectral Theorem)**

*Let $A$ be a self-adjoint on $\mathcal{H}$ with cyclic set $\{b_k\}$. Then there is a unique matrix-valued measure $\mathbf{M}$ with entries $\mathbf{M}_{i,j}$ so that*

$$\mathbf{B}^*(A - z\mathbf{I})^{-1}\mathbf{B} = \int \frac{d\mathbf{M}(t)}{t - z} \qquad \text{for } z \in \mathbb{C}\backslash\mathbb{R},$$

*i.e.* $\left((A - z\mathbf{I})^{-1} b_j, b_i\right)_{\mathcal{H}} = \int \frac{d\mathbf{M}_{i,j}(t)}{t-z}$. *Namely,* $A \sim M_t$ *on* $L^2(\mathbf{M}) = L^2(\mathbb{R}, \mathbf{M}; \mathbb{C}^d)$ *with* $\|f\|^2_{L^2(\mathbf{M})} = \int \left([d\mathbf{M}(t)]f(t), f(t)\right)_{\mathbb{C}^d}$.

We associate scalar spectral measure $\mu := \operatorname{tr} \mathbf{M}$. Then $d\mathbf{M} = W d\mu$ with $W = B^*B$, $B(t) = (\widetilde{b}_1(t), \widetilde{b}_2(t), \ldots)$, and the vector-valued integral

$$\int [d\mathbf{M}]f = \int W(t) f(t) d\mu(t).$$

# Spectral Measure of $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$ and decomposition

- The columns of $\mathbf{B}$ form a cyclic set for all $A_\Gamma$.
- So via the Spectral Theorem,

$$F_\Gamma(z) := \mathbf{B}^*(A_\Gamma - z\mathbf{I})^{-1}\mathbf{B} = \int_{\mathbb{R}} \frac{d\mathbf{M}_\Gamma(t)}{t-z},$$

  defines the family $\{\mathbf{M}_\Gamma\}$ of spectral measures of $A_\Gamma$.
- With $\mu_\Gamma := \operatorname{trace} \mathbf{M}_\Gamma$ and $W_\Gamma = B_\Gamma^* B_\Gamma$ we have

$$d\mathbf{M}_\Gamma = W_\Gamma d\mu_\Gamma.$$

Our goal is to relate $\mathbf{M}$ and $\mathbf{M}_\Gamma$ (or $\mu$ and $\mu_\Gamma$).
What of rank one pert. theory generalizes to finite rank?

- Lebesgue decomp. $d\mu = w dx + d\mu_s$, $w = d\mu/dx$ yields corresponding decomposition of $\mathbf{M}$:

$$d\mathbf{M}(x) = d\mathbf{M}_{ac}(x) + d\mathbf{M}_s(x).$$

Let $G(x) := \int_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2}$, and Cauchy transform $F_\gamma(z) := \int_{\mathbb{R}} \frac{d\mu_\gamma(t)}{t-z}$.

## Theorem (Aronszajn–Donoghue)

*When $\gamma \neq 0$, the sets*

$$S_\gamma = \left\{ x \in \mathbb{R} \middle| \lim_{y \to 0} F(x+iy) = -1/\gamma; G(x) = \infty \right\},$$

$$P_\gamma = \left\{ x \in \mathbb{R} \middle| \lim_{y \to 0} F(x+iy) = -1/\gamma; G(x) < \infty \right\},$$

$$C = \left\{ x \in \mathbb{R} \middle| \lim_{y \to 0} \operatorname{Im} F(x+iy) \neq 0 \right\}$$

*contain spectral information of $A_\gamma$ as follows:*

(i) *The sets $S_\gamma$, $P_\gamma$ and $C$ are mutually disjoint.*

(ii) *Set $P_\gamma$ is the set of eigenvalues, and set $C$ ($S_\gamma$) is a carrier for the absolutely (singular) continuous measure, respectively.*

(iii) *The singular parts of $A$ and $A_\gamma$ are mutually singular.*

- Main tool: Aronszajn–Krein formula $F_\gamma = \frac{F}{1+\gamma F}$.
- Literature provides finer results and pathological examples.

# Finite rank Kato–Rosenblum (simple proof)

On the upper half plane $F_\Gamma = (\mathbf{I} + F\Gamma)^{-1}F = F(\mathbf{I} + \Gamma F)^{-1}$.

**Theorem**

*For self-adjoint $A$, $T$ with $A \sim T$(Mod finite rank), the absolutely continuous parts of $A$ and $T$ are unitarily equivalent.*

**Theorem (Wave operators)**

*The wave operators exist, i.e. defining $\mathcal{W}^\Gamma(\tau) := e^{i\tau A_\Gamma} e^{-i\tau A} P_{\mathrm{ac}}$, where $P_{\mathrm{ac}}$ is the orth. proj. onto the absolutely continuous part of $A$, the strong limit s-lim$_{\tau\to\pm\infty}\mathcal{W}^\Gamma(\tau)$ exists.*

Idea of proof for wave operators: For any $f \in L^2(\mathbf{M}_{\mathrm{ac}})$ we have

$$\text{s-lim}_{\tau\to\pm\infty} V_\Gamma P_{\mathrm{ac}}^{A_\Gamma} \mathcal{W}^\Gamma(\tau) f = (\mathbf{I} + \Gamma F_\pm) f.$$

# Vector mutual singularity of singular parts

**Definition**

Matrix-valued measures $\mathbf{M} = W\mu$ and $\mathbf{N} = V\nu$ are *vector mutually singular* ($\mathbf{M} \perp \mathbf{N}$) if one can extent $W$ and $V$ so that

$$\operatorname{Ran} W(t) \perp \operatorname{Ran} V(t) \qquad \mu\text{-a.e. and } \nu\text{-a.e.}$$

**Theorem**

*Singular parts of the matrix-valued measures $M$ and $M^\Gamma$ satisfy*

$$\mathbf{M}_\mathrm{s} \perp \Gamma \mathbf{M}_\mathrm{s}^\Gamma \Gamma \qquad \textit{and} \qquad \mathbf{M}_\mathrm{s}^\Gamma \perp \Gamma \mathbf{M}_\mathrm{s} \Gamma.$$

The proof uses spectral representation and a matrix $A_2$ condition.

# Aleksandrov Spectral Averaging

**Theorem**

*Let $\Gamma_0$ be a self-adjoint and $\Gamma_1$ be a positive definite $d \times d$ matrix. Consider scalar-valued Borel measurable $f \in L^1(\mathbb{R})$. We have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) d\mathbf{M}_{\Gamma_0 + t\Gamma_1}(x) dt = \Gamma_1^{-1} \int_{\mathbb{R}} f(x) dx.$$

*In particular, for any Borel set $B$ with zero Lebesgue measure $\mathbf{M}_{\Gamma_0 + t\Gamma_1}(B) = \mathbf{0}$ for Lebesgue a.e. $t \in \mathbb{R}$.*

# Summary

- Spectral Theorem and matrix-valued spectral measures
- No Aronszajn–Donoghue for higher rank perturbations
- Kato–Rosenblum simple proof and existence of wave operators
- Vector mutual singularity of matrix-valued spectral measures
- Aleksandrov spectral averaging yields some mutual singularity also of scalar-valued spectral measures