Quasisymmetric Embeddings of Slit Sierpiński Carpets into $\mathbb{R}^2$

Wenbo Li

Graduate Center, CUNY

CAFT, July 21, 2018

Sierpiński Carpet

- $\mathcal{S}_3$ - the standard Sierpiński carpet in $[0,1]^2$.

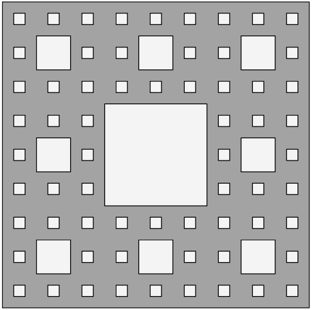


Figure: Finite generation of standard Sierpiński carpet

- Whyburn's Theorem: $X = \mathbb{S}^2 \setminus \bigcup_i^\infty D_i \overset{\text{homeo}}{\approx} \mathcal{S}_3$ if
  1. $\overline{D_i} \cap \overline{D_j} = \emptyset, \ \forall \ i \neq j.$
  2. $\mathrm{diam}(D_i) \to 0, \ \text{as } i \to \infty.$
  3. $\overline{(\bigcup_i D_i)} = \mathbb{S}^2.$
- A metric space (X, d) is a (metric) carpet if $X \overset{\text{homeo}}{\approx} \mathcal{S}_3$.

Quasiconformality and Quasisymmetry

- $f : X \longrightarrow Y$ - homeomorphism
- f is (metrically) quasiconformal if $\exists\ K \geq 1$ s.t.

$$H_f(x) = \limsup_{r \to 0} \frac{\sup\{d_Y(f(x), f(y)) : d_x(x,y) \leq r\}}{\inf\{d_Y(f(x), f(y)) : d_x(x,y) \geq r\}} \leq K$$

for all $x \in X$.

- f is quasisymmetric if $\exists$ homeomorphism $\eta : [0, \infty) \to [0, \infty)$ s.t.

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right)$$

for all $x, y, z \in X$ with $x \neq z$.

Geometry of Carpets: Questions

Question 1 Is every carpet $X \overset{\text{BL}}{\approx} \mathcal{S}_3$?
NO. Easy. $\mathcal{S}_5$. Using Hausdorff dimension.

Question 2 Is every carpet $X \overset{\text{QS}}{\approx} \mathcal{S}_3$?
NO. Hard. $\mathcal{S}_5$. (Bonk Merenkov)

Question 3 Is every carpet $X \overset{\text{QS}}{\hookrightarrow} \mathbb{R}^2$?
NO. A self-similar slit carpet. (Merenkov, Wildrick)

Question 4 Is every group boundary carpet $X \overset{\text{QS}}{\hookrightarrow} \mathbb{R}^2$?
Not known.

Kapovich-Kleiner conjecture (simplified version): Every group boundary carpet X can be quasisymmetrically embedded into $\mathbb{R}^2$.

General Problem: Characterize carpets which can be quasisymmetrically embedded into $\mathbb{R}^2$.

Our Result: We give a complete characterization for a special kind of carpets.

Slit Domains

- Dyadic slit domain of nth-generation corresponding to $\mathbf{r} = \{r_i\}_{i=0}^{\infty}$:

$$\mathcal{S}_n(\mathbf{r}) = [0,1]^2 \setminus \left(\bigcup_{i=0}^{n} \bigcup_{j=1}^{2^i} s_{ij} \right),$$

  - $s_{ij} \subset \Delta_{ij}$ where Δ_{ij} is a dyadic square of generation i
  - The center of s_{ij} coincides with the center of Δ_{ij}
  - $l(s_{ij_1}) = l(s_{ij_2}) = r_i \cdot \frac{1}{2^i}$ for $j_1, j_2 \in \{1, \ldots, 2^i\}$.

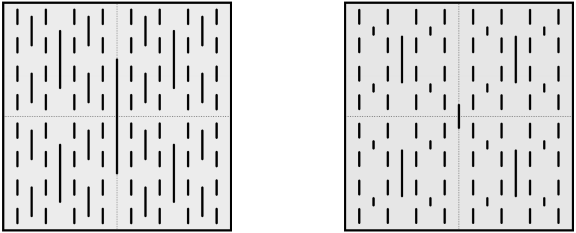


Figure: Slit domains corresponding to $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{10}, \frac{2}{5}, \frac{1}{8}, \frac{1}{2})$

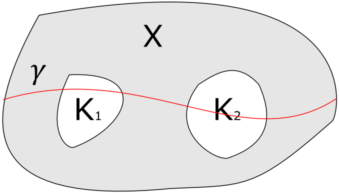
Slit Sierpiński Carpets

- Let $\mathbf{r} = \{r_i\}_{i=0}^{\infty}$ be a sequence satisfying:
  1. $r_i \in (0,1), \forall i \in \mathbb{N}$
  2. $\limsup_{i\to\infty} r_i < 1$
- $\mathscr{S}_n(\mathbf{r}) = \overline{\mathcal{S}_n(\mathbf{r})}$ equipped with the path metric.
- The limit

$$\mathscr{S}(\mathbf{r}) = \underset{\text{G-H limit}}{\lim_{n\to\infty}} \mathscr{S}_n(\mathbf{r})$$

  is called a dyadic slit Sierpiński carpet corresponding to $\mathbf{r}$.
- $\mathcal{S}(\mathbf{r}) = [0,1]^2 \setminus \left(\bigcup_{i=0}^{\infty} \bigcup_{j=1}^{2^i} s_{ij} \right)$.
- $\mathscr{S}(\mathbf{r})$ is a metric carpet.

Transboundary Modulus



- Let Γ be a family of curves in metric measure space (X, d, μ). The transboundary 2-modulus of Γ with respect to $\{K_i\}$ is defined as

$$\text{tr-mod}_{X,\{K_i\}}(\Gamma) = \inf\left\{\int_{X\setminus\bigcup_i K_i} \rho^2 d\mu + \sum_{i\in I} \rho_i^2\right\}$$

where infimum is taken over all Borel function $\rho : X\setminus\bigcup_i K_i \to (0, \infty]$, and weights $\rho_i \geq 0$ such that

$$\int_{\gamma\cap X\setminus\bigcup_i K_i} \rho\ ds + \sum_{\gamma\cap K_i\neq\phi} \rho_i \geq 1, \ \ \forall\gamma\in\Gamma.$$

- Transboundary n-modulus is a conformal invariant and quasiconformal quasi-invariant on $\mathbb{R}^n$.

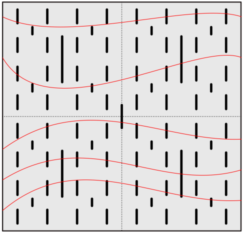
Transboundary Modulus Estimation

Theorem 1 (Upper bound, Hakobyan-Li, 2017)

Let $\Gamma = \Gamma(L, R, \mathbb{I})$ *and* $\mathcal{K}_n$ *be the collection of all slits in* $\mathcal{S}_n(\mathbf{r})$*. Then*

$$tr\text{-}mod_{\,\mathbb{I},\mathcal{K}_n}(\Gamma) \leq \prod_{i=0}^{n}(1 - \epsilon r_i^2) + 3\epsilon$$

for $\forall\ \epsilon$ *small enough.*

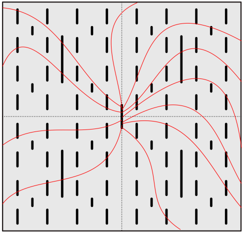


Transboundary Modulus Estimation

Lemma 2

Let $\Gamma_{io} = \Gamma(s_0, \partial\mathbb{I}, \mathbb{I}\backslash s_0)$. *If* $\{r_i\} \notin \ell^2$, *then*

$$\lim_{n\to\infty} tr\text{-}mod_{\mathbb{I}\backslash s_0, \mathcal{K}_n\backslash\{s_0\}}(\Gamma_{io}) = 0.$$



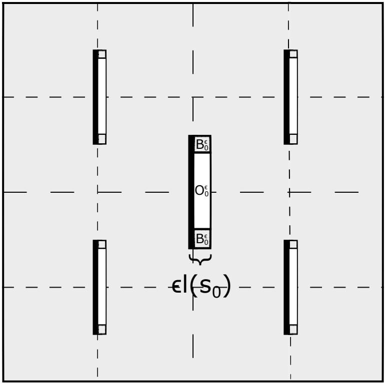


Figure: Slit collar

$\rho_n^\epsilon(x) = \chi_{\mathbf{B}_n^\epsilon \cup \mathbf{R}_n^\epsilon}(x) = \chi_{\mathbb{I}\setminus \mathbf{O}_n^\epsilon}(x)$

$A(\rho, \rho_i) \leq \prod_{i=0}^{n}(1 - \epsilon r_i^2) + 3\epsilon.$

$$\rho_n^j = \begin{cases} \epsilon l(v_j), & v_j \text{ are chosen slits} \\ 0 \quad , & \text{otherwise} \end{cases}$$

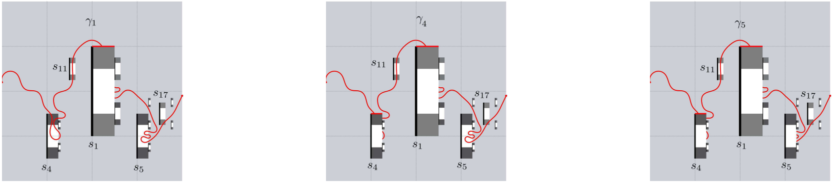


Figure: admissible

- Replacing each curve with a new one.
- the new curve is in $\mathbb{I} \setminus \mathbf{O}_n^{\epsilon}$.
- the new curve is "shorter" than the old one.

Main Theorem

Main Theorem (Hakobyan-Li, 2017)

$\mathscr{S}(\mathbf{r})$ can be quasisymmetrically embedded into $\mathbb{R}^2$ if and only if $\mathbf{r} = \{r_i\}_{i=0}^{\infty} \in \ell^2$.

$\mathbf{r} \notin \ell^2 \Longrightarrow \nexists \varphi : \mathscr{S}(\mathbf{r}) \hookrightarrow \mathbb{R}^2$ quasisymmetrically

1. Suppose not, i.e. $\exists\ \varphi : \mathscr{S} \hookrightarrow \mathbb{R}^2$, where φ is η-QS.
2. $\exists f_n : \mathcal{S}_n \to \mathbb{R}^2$ is quasiconformal.

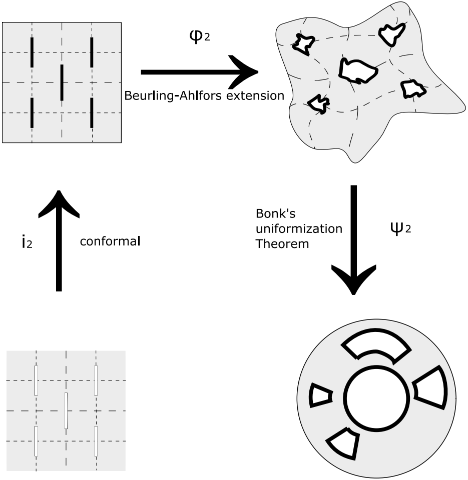


Figure: An illustration of $f_2 = \psi_2 \circ \varphi_2 \circ i_2$.

$\mathbf{r} \notin \ell^2 \Longrightarrow \nexists \varphi : \mathscr{S}(\mathbf{r}) \hookrightarrow \mathbb{R}^2$ quasisymmetrically

3. If $\{r_i\} \notin \ell^2$ then $\text{tr-mod}(\Gamma_{io}) \to 0$ as $n \to \infty$.(Lemma 2)
4. $\text{tr-mod}(f_n(\Gamma_{io})) = \frac{2\pi}{\log \frac{R_n}{r_n}} > \epsilon > 0$, since quasisymmetry on inner and outer boundary.
5. $0 < \epsilon < \text{tr-mod}(f_n(\Gamma_{io})) \leq C \cdot \text{tr-mod}(\Gamma_{io}) \to 0.$ Contradiction.

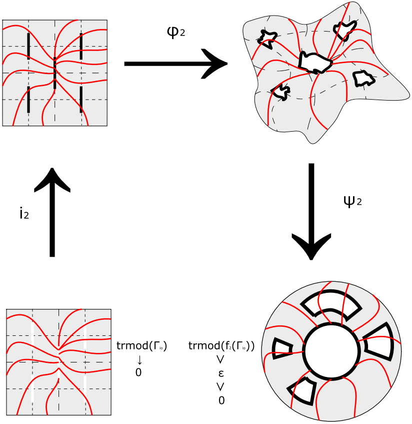


Figure: An illustration of the proof.

Corollary 3

There exists a doubling, linearly locally contractible metric space X which is homeomorphic to $\mathbb{R}^2$ or $\mathbb{S}^2$ such that

- *X is not quasisymmetrically embedded into $\mathbb{R}^2$ or $\mathbb{S}^2$, respectively.*
- *Every weak tangent of X is quasisymmetric equivalent to $\mathbb{R}^2$ with a uniformly bounded distortion function.*

Further Questions and Extensions

- What is a general slit carpet?
  A metric carpet which is the closure of a slit domain equipped with path metric and uniformly relatively separated.

- What is the equivalent criterion of planar quasisymmetric embeddability for general slit carpets?
  One guess is that the length of slits in each ball should be ℓ^2-uniformly relatively bounded.

- Are the statements true for higher dimensions?
  A similar statement for necessity is valid for higher dimensions. The sufficiency is not known.
  A similar statement for corollary 3 in higher dimensions is also valid and is working in progress.

Thank you.