Algebra & PDE : 3 tales.

I. (i) Conjecture (O. Hesse, 1859). Let $u(x_1,\dots,x_N)$ be a homogeneous polynomial of $N>1$ variables. Then, $\text{Hess } u = \left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)_{j,k=1}^{N} \equiv 0 \Rightarrow$

$\frac{\partial u}{\partial x_j}$, $j=1,\dots,N$ are lin. dependent $\Leftrightarrow$

$\nabla u : \mathbb{C}^n \to$ hyperplane.

(ii) $N=2$, $u(x,y)$ is homogeneous of degree $k+1$.

$u_{yy}\cdot u_{xx} - u_{xy}^2 = 0$. $u_x = f$, $u_y = g$; f,g are homogeneous of deg k.

$$f_x g_y - f_y g_x = 0 \Rightarrow \frac{f_x}{g_x} = \frac{f_y}{g_y} =: \lambda \quad (1)$$

$$x f_x + y f_y = kf;\ x g_x + y g_y = kg = \frac{1}{\lambda} x f_x +$$

$$+ \frac{1}{\lambda} y f_y = \frac{k}{\lambda} f;\quad \text{So, } f = \lambda g;$$

$$\therefore f_x = \lambda_x g + \lambda g_x \overset{(1)}{=} \lambda g_x.\quad \lambda_x = \lambda_y = 0,\ \lambda = c.$$

$u_x = c u_y \therefore \nabla u : \mathbb{C}^2 \to$ line #

<u>P. Gordan & M. Noether (1876)</u> - Hesse C. is true for $N=2,3,4$

False for $N \geq 5$: (Gordan & Noether)

Exemple:

deg 3: $f(x_1, \dots, x_5) = x_1 x_4^2 + x_2 x_4 x_5 + x_3 x_5^2$

$D_1 f = x_4^2$, $D_2 f = x_4 x_5$, $D_3 f = x_5^2 \Rightarrow$

$(D_1 f)(D_3 f) - (D_2 f)^2 \equiv 0 \Rightarrow \mathrm{Hess}_f \equiv 0$

since $\nabla f : \mathbb{C}^n \to \{ x_1 x_3 - x_2^2 = 0 \}$.

Note: $D_1 D_3 f - D_2^2 f \equiv 0$, f satisfies <u>linear</u> PDE

(iii) Looking for the higher ground:

$F : \mathbb{C}^N \to \mathbb{C}$, a meromorphic, "purely nonlinear".

$F \circ \nabla u = 0 \not\Rightarrow u$ is linear?

Not true in general (Gordan & Noether)

<u>Q</u>. Anything is true in this generality?

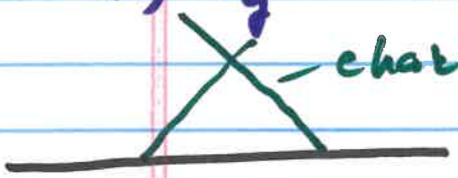
Examples: (a) $N=2$, $F = x^2 + y^2 - 1$, $u_x^2 + u_y^2 - 1 = 0$ (the eiconal equation of light rays), u is entire $\Rightarrow$ u is linear (D. Kh. -'95).

(b) "The right Theorem: (J. Guerra, '97). $F : \mathbb{C}^2 \to \mathbb{C}$, F is meromorphic, purely nonlinear, $F(u_x, u_y) = 0$ and u is meromorphic $\Rightarrow$ u is a linear function.

So, u is a meromorphic solution of $F(u_x, u_y) = 0$, F is meromorphic near $(0,0)$, nonlinear.
$\therefore$ u is linear.

"Moral of the story": Nonlinearity causes branching singularities.

Reason: 1. u_x, u_y are constants on characteristics

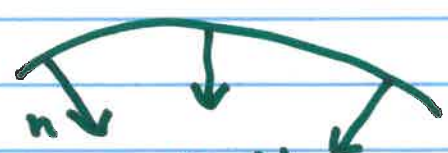


2. Nonlinearity $\Rightarrow$ different slopes $\Rightarrow$ multi-valuedness.

Another moral: number of variables matter!

(*) Ex $N = 3$, $u = z + \varphi(x + iy)$, φ = anything
$$u_x^2 + u_y^2 + u_z^2 \equiv 1.$$

Yet... in $\mathbb{R}^N$: $u \in C^1(\mathbb{R}^N)$, $\sum_1^N u_{x_j}^2 \equiv 1 \Rightarrow$ u = linear.



(*) indicates that for larger N there are more opportunities for polynomial solutions of, say, eiconal...

$$u = f(z_1 + i z_2) + g(z_3 + i z_4), \quad \sum_1^4 u_{z_j}^2 \equiv 0 \ldots$$

(II) Algebras of solutions of linear PDE.

(i) Example: $P = \sum_1^N z_i^2$, i.e., $P(D) = \Delta$.

If $\Delta u = 0$, and $\Delta u^2 = 0 \Rightarrow$

$$\Delta u^2 = 2u\underbrace{\Delta u}_{0} + 2(\nabla u)^2 = 0$$

$$\Rightarrow \Delta u^3 = \underbrace{\Delta u^2}_{0} \cdot u + \underbrace{u\,\Delta u^2}_{0} + 2u(\nabla u)^2 = 0, \text{ etc.}$$

$\sum_1^N u_{z_i}^2$

$N=2$, $\sum_1^2 \left(\frac{\partial u}{\partial z_i}\right)^2 = 0 \Leftrightarrow \left(\frac{\partial u}{\partial z_1} + i\frac{\partial u}{\partial z_2}\right)\left(\frac{\partial u}{\partial z_1} - i\frac{\partial u}{\partial z_2}\right) = 0$, $\Rightarrow$ u is either analytic, or antianalytic

B. I. Korenblum in late 70's Conjectured that if $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^3$ is a domain, and $\Delta u = \Delta u^2 = 0 (\Rightarrow 0 = \Delta u^k, k \in \mathbb{N})$, then, up to an appropriate rotation, u must be either analytic, or anti-analytic function in 2D.

B.I. produced several "proofs" of the statement all with gaps.

There is a reason: The conjecture is false as stated. (There are whole websites dedicated to so-called "harmonic morphisms" containing lots of counterexamples.).

what is true?

<u>Theorem</u> (DK, ~92, unpublished). If u is a polynomial in $\mathbb{R}^3$, $\Delta u = \Delta u^2 = 0$, then after an appropriate rotation, u must be an analytic, or antianalytic function in 2D.

<u>Main idea</u> $u = u_0 + \dots + u_m$, u_j are homogeneous polynomials. Then, u_m satisfies $(\nabla u_m)^2 = 0$.

<u>Theorem (DK, ~92, unpublished)</u> Let P be a homogeneous polynomial in $\mathbb{C}^3$, and $\{P=0\} =: \Gamma$ is characteristic wrt Δ, i.e., $\sum_1^3 \left(\frac{\partial P}{\partial z_j}\right)^2 = 0$ on Γ. Then, $\Gamma = \{z_1^2 + z_2^2 + z_3^2 = 0\}$ or $\Gamma = \{\sum_1^3 \alpha_j z_j = 0, \sum_1^3 \alpha_j^2 = 0\}$

Thus, $u_m = (\Sigma_1^3 \alpha_j z_j)^m$, $\Sigma \alpha_j^2 = 0$.

Recap : (a) $u(z_1, z_2, z_3)$, a homogenous polynomial in $\mathbb{C}^3$: $\Delta u = \Delta u^2 = 0 \Rightarrow u = \left(\sum_1^3 \alpha_j z_j\right)^m$, $\sum_1^3 \alpha_j^2 = 0$

(b) In view of the results on eiconal in 2 variables, Korenblum's conjecture holds for entire u in $\mathbb{C}^3$

$\left(\text{on } \{u=0\}, \ \left(\frac{\partial u}{\partial z_1}\right)^2 + \left(\frac{\partial u}{\partial z_2}\right)^2 + \left(\frac{\partial u}{\partial z_3}\right)^2 = 0\right.$

and writing on $\{u=0\}$, $z_3 = \varphi(z_1, z_2)$

we have
$$(\varphi_{z_1})^2 + (\varphi_{z_2})^2 = -1, \text{ an eiconal.}$$

(c) With appropriate modification, Korenblum's conjecture holds for polynomials in N-variables, but the statement must be adjusted, (and not pretty anymore):

Ex. $u = f(x+iy) + g(t-is)$ in $\mathbb{C}^4(x,y,t,s)$.

$\Delta u = \Delta u^2 = 0, \ldots$ etc.

(iii) B. Shekhtman - T. McKinley 's Theorem ('17)

Recall Gordan-Noether example :

$f = x_1 x_4^2 + x_2 x_4 x_5 + x_3 x_5^2$, $(D_1 f)(D_3 f) - (D_2 f)^2 = 0$

Also $D_1 D_3 f - D_2^2 f \equiv 0$.

Conjecture ('17, Shekhtman–McKinley):

Let P, f be homogeneous polynomials,

$P(D_1 f, \dots, D_N f) = 0 \Rightarrow P(D) f = 0$.

<u>Ex</u> $\sum_1^3 \left(\frac{\partial f}{\partial z_j}\right)^2 = 0 \Rightarrow \underbrace{f = \left(\sum_1^3 \alpha_j z_j\right)^m}_{f\text{-homogeneous}}, \sum_1^3 \alpha_j^2 = 0$

$\Rightarrow \Delta f = 0$.

"Converse" to the Conjecture ('17, McKinley–Sh)

If P is homogeneous, $P(D)[f^k] = 0$, $k=1,3$.

then $P(D_1 f, \dots, D_N f) = 0$.

<u>Cor.</u> Let f be homogeneous polynomial.

$\mathcal{P}(f) = \text{span}\{f(\cdot + b)^k : b \in \mathbb{C}^N, k \in \mathbb{N}\}$.

If $\mathcal{P}(f) \neq \mathbb{C}[z_1, \dots, z_N]$, then $\text{Hess} f \equiv 0$.

The reason:

(A. Pinkus – B. Wajnryb) '95): TFAE:

(i) $\mathcal{P}(f) \neq \mathbb{C}[z_1, \dots, z_N)$

(ii) $\exists$ <u>polynomial</u> p: $p(D)[f^k] = 0$, $k \in \mathbb{N}$

(iii) $\mathcal{P}(f) \neq C(\mathbb{C}^N)$.

J'accuse: "Wrong Theorem".

<u>Theorem (DK, '18).</u> Let $P(D) := \sum_{|\alpha| \le m} a_\alpha(z) \partial^\alpha$,
$z = (z_1, \dots, z_N)$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\partial^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_N}\right)^{\alpha_N}$,
be a linear DO with entire coefficients.
Let u be entire, $P(D)[u^k] = 0$, for all k in some arithmetic progression, e.g., $k = 2n+1$, $n \in \mathbb{N}$. Then, $\operatorname{Hess} u \equiv 0$.
More precisely, $\boxed{\sum_{|\alpha| = m} a_\alpha (\nabla u)^\alpha \equiv 0.}$ (*)

Sh-McK follows, $P = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$, constant coefficients DO.

"Proof" (sketch): For any $c \in \mathbb{C}$ write locally the expansion $\frac{1}{u-c} = \sum_0^\infty c_k u^k$.
(WLOG, k runs over $\mathbb{N}$).
HYPOTHESIS $\Rightarrow$ $P(D)\left[\frac{1}{u-c}\right] = 0$ (locally)
$\Rightarrow$ globally in $\mathbb{C}^N \setminus \{u = c\}$.

<u>Theorem</u> (Delassus - Le Roy). If U satisfies $P(D)(U) = 0$ in a neighborhood of a hyper surface Γ ($\{u=c\}$), Γ must be everywhere characteristic provided that U is singular on Γ.

Recap: $-\sum_{|\alpha|\le m} a_\alpha(z)\,\partial^\alpha[u^k] = 0$, u - entire, k runs over an arithmetic progression, then

$$\sum_{|\alpha|=m} a_\alpha(z)\,(\nabla u)^\alpha \equiv 0, \Rightarrow \mathrm{Hess}\,u \equiv 0.$$

- Delassus - Le Roy says simply, that singularities of a solution of linear PDE "propagate" in $\mathbb{C}^N$ along characteristic surfaces.

- Instead of functions $\left\{\frac{1}{u-c}\right\}$ one can, of course, take any continual family $\{F_s\}$ $F_s := \sum_0^\infty c_s z^{n_s}$, such that each F_s has only isolated singularities (poles) on the circle of convergence. Then, k for u^k runs over $\{n_s\}$, e.g.,

$$\frac{1}{z^3-1} = \sum_0^\infty z^{3k}, \ldots$$

- A modern proof of Del-LR theorem is based on a simple extension of C-K theorem due to

(III) Dirichlet's Problem $\rightarrow$ Goursat's Problem $=$ Interpolation problem $\rightarrow$ Division problem.

(i) Let $\Omega \subset \mathbb{R}^N$, $\partial\Omega =: \Gamma = \{\varphi(x) = 0, \varphi$ - polynomial$\}$, f = polynomial, or entire.

Dirichlet Problem: find u:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \Gamma \end{cases}$$

$\rightarrow$ $u = f + q\varphi$, q - real analytic in Ω

$\Delta u = 0 \iff$ find q: $\Delta(q\varphi) = -\Delta f$.

$\rightarrow$ Interpolate f on $\Gamma = \{\varphi = 0\}$ by u: $\Delta u = 0$.

$\rightarrow$ If $P = q.c.d.(f, \varphi)$, $P \mid u$.

<u>Q.</u> Which polynomials divide harmonic functions?

Ex. (M. Brelot - G. Choquet, '55).

<u>Thm</u>: Let $P \geq 0$ on $\mathbb{R}^n$, u is a harmonic polynomial. If $u \in (P)$ = ideal generated by P, $u \equiv 0$.

Sketch: WLOG, u, P are homogeneous. If $u = Pq$, then on the unit sphere S^{n-1}, $Pq \perp q$ in L^2. $\therefore Pq^2 \equiv 0$, $\therefore Pq = u \equiv 0$.

<u>Q</u>: On which varieties can harmonic functions vanish?

<u>Ex</u>. The ideal $(x^2 - y)$ doesn't contain harmonic functions (L. Flatto - D. Newman - H. S. Shapiro, '64). Moreover, $\{y = x^2\}$ cannot be a zero set of an entire harmonic functions

- Cones $\{z^2 = \alpha(x^2 + y^2)\}$ are settled by D. Armitage

Let $\Omega := \Big\{ \underbrace{\sum_{1}^{N} \frac{x_i^2}{a_i^2} - 1}_{\varphi(x)} \le 0 \Big\}$, an ellipsoid.

<u>Thm</u> (DK–H. S. Shapiro, '92). For every entire function f, $\exists!\, u : \Delta u = 0$, u - entire and $u = f$ on $\partial\Omega = \Gamma := \{\varphi = 0\}$.

In other words,

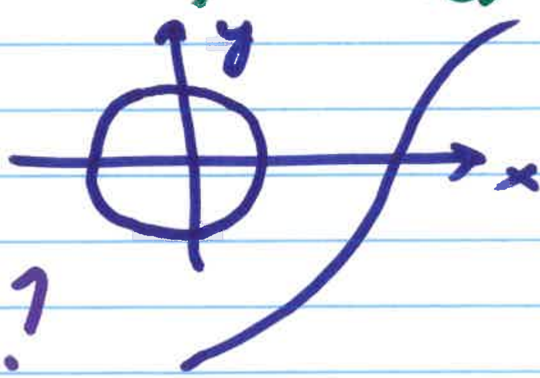
$$(*) \quad f = u + q\varphi, \quad \Delta u = 0, \quad f, u, q \text{ - entire.}$$

<u>Conjecture</u> $\nexists$ other polynomials φ : $(*)$ holds. (DK-HSS, '92)

Thm (H. Render, '06) Conjecture is true for all $\varphi = \sum_{0}^{m} \underbrace{\varphi_k}_{\text{homogeneous}}$: φ_m is elliptic.

<u>Yet</u>: What if $\varphi(x,y) = x^2 + y^2 - 1 + \varepsilon x^{2m+1}$?

$m=1$



True, Lundberg–Render, 2011

$m=2$???

<u>Q.</u> In the principal ideal $(x^2+y^2-2z^2)$ are there any more harmonic polynomials?

Mathematical Surveys and Monographs

Volume [illegible]

Linear Holomorphic Partial Differential Equations and Classical Potential Theory

Dmitry Khavinson
Erik Lundberg

AMERICAN MATHEMATICAL SOCIETY

Why do solutions of linear analytic PDE suddenly break down? What is the source of these mysterious singularities, and how do they propagate? Is there a mean value property for harmonic functions in ellipsoids similar to that for balls? Is there a reflection principle for harmonic functions in higher dimensions similar to the Schwarz reflection principle in the plane? How far outside of their natural domains can solutions of the Dirichlet problem be extended? Where do the continued solutions become singular and why?

This book invites graduate students and young analysts to explore these and many other intriguing questions that lead to beautiful results illustrating a nice interplay between parts of modern analysis and themes in "physical" mathematics of the nineteenth century. To make the book accessible to a wide audience including students, the authors do not assume expertise in the theory of holomorphic PDE, and most of the book is accessible to anyone familiar with multivariable calculus and some basics in complex analysis and differential equations.

Thank You!

ISBN 978-1-4704-3780-0

9 781470 437800

SURV/232

For additional information and updates on this book, visit

www.ams.org/bookpages/surv-232

AMS

www.ams.org