Complex-analytic and other properties of the generalized hypergeometric functions and their ratios

Dmitrii Karp (joint work with Elena Prilepkina, José Luis López, Alexander Dyachenko and Sergei Kalmykov)
dmkrp.wordpress.com

Far Eastern Federal University

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Generalized hypergeometric function: questions

Set $\mathbf{a} = (a_1, a_2, \ldots, a_p) \in \mathbb{C}^p$, $\mathbf{b} = (b_1, b_2, \ldots, b_q) \in \mathbb{C}^q$. Then

$$ {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = {}_pF_q(\mathbf{a};\mathbf{b};z) := \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!} z^n, $$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the rising factorial. The series converges for all $z \in \mathbb{C}$ if $p \le q$ and for $|z| < 1$ if $p = q+1$.

- Analytic continuation of $z \to {}_pF_{p-1}(z)$ to $|z| \ge 1$
- Analytic continuation of $(\mathbf{a}, \mathbf{b}) \to {}_pF_{p-1}(1)$ to $\Re[\sum_k a_k - \sum_j b_j] > 0$ (${}_pF_{p-1}(1)$ is important in physics)
- Geometric properties of $z \to {}_pF_{p-1}(z)$ (univalence, starlikeness, convexity etc.) and ratios
- Values of $z \to {}_pF_{p-1}(z)$ on the banks of the branch cut $[1, \infty)$
- Bounds for $z \to {}_pF_{p-1}(z)$ in the complex plane
- Location of zeros of entire functions ${}_pF_q$, $p \le q$ (reality of zeros, zero-free regions, etc.)

Important ingredient: Meijer's G-function

Definition of Meijer's G-function (Meijer, around 1940)

Suppose $0 \le m \le q$, $0 \le n \le p$ are integers, $\mathbf{a} = (a_1, a_2, \dots, a_p) \in \mathbb{C}^p$, $\mathbf{b} = (b_1, b_2, \dots, b_q) \in \mathbb{C}^q$ are such that $a_i - b_j \notin \mathbb{N}$ for $i = 1, \dots, n$, $j = 1, \dots, m$. Define

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) :=$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}} \underbrace{\frac{\Gamma(b_1+s)\cdots\Gamma(b_m+s)\Gamma(1-a_1-s)\cdots\Gamma(1-a_n-s)}{\Gamma(a_{n+1}+s)\cdots\Gamma(a_p+s)\Gamma(1-b_{m+1}-s)\cdots\Gamma(1-b_q-s)}}_{\mathcal{G}(s)} z^{-s} ds.$$

The contour $\mathcal{L}$ begins and ends at infinity and separates the poles $-b_j - k$, $k = 0, 1, \dots$ from the poles $1 - a_i + l$, $l = 0, 1, \dots$

Important ingredient: Meijer's G-function

Mostly, we need only a particular case (Meijer-Nørlund function):

$$G^{p,0}_{q,p}\left(z \,\middle|\, \begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) := \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)} z^{-s} ds.$$

The contour $\mathcal{L}$ is a vertical line on the right of all the poles of the integrand or the left loop beginning and ending at $-\infty$ and leaving all the poles on the left.

Notation:

$$\Gamma(\mathbf{a}) = \Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p), \quad (\mathbf{a})_n = (a_1)_n(a_2)_n\cdots(a_p)_n,$$
$$\mathbf{a}+\mu = (a_1+\mu, a_2+\mu, \ldots, a_p+\mu);$$

in particular, $(\mathbf{a}) = (\mathbf{a})_1 = a_1\cdots a_p$; inequalities like $\Re(\mathbf{a}) > 0$ and properties like $-\mathbf{a} \notin \mathbb{N}_0$ will be understood element-wise. The symbol $\mathbf{a}_{[k]}$ stands for the vector $\mathbf{a}$ with omitted k-th element.

Key tool: integral representations

Termwise integration leads to the Laplace transform representations

$$ {}_{p+1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^{\infty}e^{-zt}G^{p+1,0}_{p,p+1}\left(t\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t}, $$

$$ {}_{p}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^{1}e^{-zt}G^{p,0}_{p,p}\left(t\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t}, $$

the generalized Stieltjes transform representation

$$ {}_{p+1}F_p\left(\begin{matrix}\sigma,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right)=\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^{1}G^{p,0}_{p,p}\left(t\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right)\frac{dt}{t(1+zt)^{\sigma}} $$

and the cosine Fourier transform representation

$$ {}_{p-1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z^2/4\right)=\frac{2\Gamma(\mathbf{b})}{\sqrt{\pi}\Gamma(\mathbf{a})}\int_0^{1}\cos(zt)G^{p,0}_{p,p}\left(t^2\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a},1/2\end{matrix}\right)\frac{dt}{t}. $$

These hold if $\Re(\mathbf{a})>0$ and also $\sum_{i=1}^{p}\Re(b_i-a_i)>0$ in the second and third formulas or $\sum_{i=1}^{p}\Re(b_i)-\sum_{i=1}^{p-1}\Re(a_i)>1/2$ in the last formula.

Extended integral representations with atom

First appearance: 1994 book by Virginia Kiryakova (derived by consecutive fractional integrations). We relaxed the restrictions on parameters and further extended these formulas to zero parametric excess $\sum_{i=1}^{p}(b_i - a_i) = 0$ as follows

$$ {}_pF_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|-z\right) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1 e^{-zt}\left\{G_{p,p}^{p,0}\left(t\,\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) + \delta_1\right\}\frac{dt}{t}, $$

$$ {}_{p+1}F_p\left(\begin{matrix}\sigma,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|-z\right) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}\int_0^1 \left\{G_{p,p}^{p,0}\left(t\,\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a}\end{matrix}\right) + \delta_1\right\}\frac{dt}{t(1+zt)^{\sigma}}, $$

$$ {}_{p-1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|-z^2/4\right) = \frac{2\Gamma(\mathbf{b})}{\sqrt{\pi}\Gamma(\mathbf{a})}\times \int_0^1 \cos(zt)\left\{G_{p,p}^{p,0}\left(t^2\,\middle|\begin{matrix}\mathbf{b}\\ \mathbf{a},1/2\end{matrix}\right) + \delta_1\right\}\frac{dt}{t}, $$

where δ_1 denotes the unit mass at the point $t = 1$ and $\sum_{i=1}^{p} b_i - \sum_{i=1}^{p-1} a_i = 1/2$ in the last formula.

Positivity of G-function

Proposition (K.-Prilepkina, 2012)

Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$ and $v_{\mathbf{a},\mathbf{b}}(t) = \sum_{j=1}^{p}(t^{a_j} - t^{b_j}) \geq 0$. Then

$$G_{p,p}^{p,0}\left(t \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) \geq 0$$

on $(0,1)$ and

$$G_{p,p+1}^{p+1,0}\left(t \,\middle|\, \begin{matrix} \mathbf{b} \\ \sigma, \mathbf{a} \end{matrix}\right) \geq 0$$

on $(0,\infty)$ for any $\sigma > 0$. In fact, more is true: if also $\mathbf{a}, \mathbf{b} > 0$ then

$$\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})} G_{p,p}^{p,0}\left(e^{-t} \,\middle|\, \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) dt$$

is infinitely divisible probability distribution on $[0,\infty)$.

Sufficient condition: weak majorization

Observation (Alzer (1997) based on Tomić (1949)): $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0,1]$ if

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_p, \quad 0 \leq b_1 \leq b_2 \leq \cdots \leq b_p,$$

$$\text{and } \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \text{ for } k = 1, 2 \ldots, p.$$

These inequalities are known as weak supermajorization and are abbreviated as $\mathbf{b} \prec^W \mathbf{a}$, where $\mathbf{a} = (a_1, \ldots, a_p)$, $\mathbf{b} = (b_1, \ldots, b_p)$.

Markov representation

Definition: Markov functions

Define $\mathcal{T}$ to be the class of functions f representable by

$$f(z) = \int_0^1 \frac{d\mu(t)}{1 - zt}$$

for some probability measure μ on $[0, 1]$. Functions $f \in \mathcal{T}$ are generating functions of the Hausdorff moment sequences.

Theorem (K.-Prilepkina, 2012)

Suppose $0 < \sigma \leq 1$, $\mathbf{a} > 0$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0, 1]$ (in particular, it suffices that $\mathbf{b} \prec^{W} \mathbf{a}$). Then ${}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; z) \in \mathcal{T}$ and the representing measure is given by

$$d\mu(t) = \frac{\Gamma(\mathbf{b})}{\Gamma(\sigma)\Gamma(\mathbf{a})} G^{p+1,0}_{p+1,p+1}\left(t \,\middle|\, \begin{matrix} 1, \mathbf{b} \\ \sigma, \mathbf{a} \end{matrix}\right) \frac{dt}{t}$$

if $\sum(b_k - a_k) > 0$ or $d\mu_1(t) = d\mu(t) + \delta_1$ if $\sum(b_k - a_k) = 0$.

Corollary 1 (K.-Prilepkina, 2012)

Suppose $0 < \sigma \leq 1$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0,1]$. Then the functions

$$z \to {}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; z) \text{ and } z \to z{}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; z)$$

are univalent in the half-plane $\Re(z) < 1$. The second function is starlike is the disk $|z| < r^*$, where $r^* = \sqrt{13\sqrt{13} - 46} \approx 0,934$.

Corollary 2 (K.-Prilepkina, 2012)

Suppose $0 < \sigma \leq 2$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0,1]$. Then the function $z \to z{}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; z)$ is univalent in the disk $|z| < r_s := \sqrt{\sqrt{32} - 5} \approx 0.81$.

Corollary 3 (K.-Prilepkina, 2012)

Suppose $\sigma \geq 1$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0,1]$. Then the function ${}_{p+1}F_p(\sigma, \mathbf{a}; \mathbf{b}; -z)$ maps the sector $0 < \arg(z) < \pi/\sigma$ into the lower half-plane $\Im(z) < 0$.

Inequalities in the left half-plane

Theorem (K.-Prilepkina, 2017)

Suppose $\mathbf{a}, \mathbf{b} > 0$ are such that $0 < a_1 \le 1$ and $v_{\mathbf{a}_{[1]},\mathbf{b}}(t) \ge 0$ on $[0,1]$ (where $\mathbf{a}_{[1]} = (a_2, \ldots, a_{p+1})$). Then the following inequalities hold in the half plane $\Re(z) < 1$:

$$\frac{2(\mathbf{a})|z-2||z|}{(\mathbf{b})(|z-2|+|z|)^2} \le |{}_{p+1}F_p(\mathbf{a};\mathbf{b};z) - 1| \le \frac{2(\mathbf{a})|z-2||z|}{(\mathbf{b})(|z-2|-|z|)^2},$$

$$\frac{4(|z-2|-|z|)}{(|z-2|+|z|)^3} \le |{}_{p+1}F_p(\mathbf{a}+1;\mathbf{b}+1;z)| \le \frac{4(|z-2|+|z|)}{((|z-2|-|z|)^3},$$

$$\frac{2(\mathbf{b})(|z-2|-|z|)}{(\mathbf{a})|z||z-2|(|z-2|+|z|)} \le \left|\frac{{}_{p+1}F_p(\mathbf{a}+1;\mathbf{b}+1;z) - 1}{{}_{p+1}F_p(\mathbf{a};\mathbf{b};z)}\right| \le \frac{2(\mathbf{b})(|z-2|+|z|)}{(\mathbf{a})|z||z-2|(|z-2|-|z|)}.$$

Jump and average value on the branch cut

Theorem (K.-Prilepkina, 2017)

Suppose $x>1$ and $\mathbf{a},\mathbf{b}$ are real vectors. Then the following identities hold true

$$
{}_{p+1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,x+i0\right)-{}_{p+1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\,\middle|\,x-i0\right)
=2\pi i\frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}G^{p+1,0}_{p+1,p+1}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1,\mathbf{b}\\ \mathbf{a}\end{matrix}\right)
$$

and

$$
\frac{{}_{p+1}F_p\left(\mathbf{a};\mathbf{b};x+i0\right)+{}_{p+1}F_p\left(\mathbf{a};\mathbf{b};x-i0\right)}{2}
=-\frac{\pi\Gamma(\mathbf{b})}{\sqrt{x}\Gamma(\mathbf{a})}G^{p+1,1}_{p+2,p+2}\left(\frac{1}{x}\,\middle|\,\begin{matrix}1/2,1,\mathbf{b}-1/2\\ \mathbf{a}-1/2,1\end{matrix}\right)
$$

Ratios of the Gauss functions

Gauss continued fraction + Markov and Stieltjes integral representations:

$$f_1(z) = \frac{{}_2F_1(a_1, a_2 + 1; b_1 + 1; z)}{{}_2F_1(a_1, a_2; b_1; z)} \in \mathcal{T}$$

for $0 \le a_1 \le b_1$, $0 \le a_2 \le b_1$. Explicit expression for the measure (under additional restriction $b_1 \ge 1$) - Belevitch (1984):

$$f_1(z) = A(a_1, a_2, b_1) + B(a_1, a_2, b_1) \int\limits_0^1 \frac{t^{a_1+a_2-1}(1-t)^{b_1-a_1-a_2} dt}{(1-zt)|{}_2F_1(a_1, a_2; b_1; 1/t)|^2},$$

where

$$B(a_1, a_2, b_1) = \frac{\Gamma(b_1)\Gamma(b_1+1)}{\Gamma(a_1)\Gamma(a_2+1)\Gamma(b_1-a_2)\Gamma(b_1-a_1+1)}$$

$$A(a_1, a_2, b_1) = 0 \text{ if } a_2 \le a_1; \qquad A = \frac{b_1(a_2-a_1)}{a_2(b_1-a_1)} \text{ if } a_2 > a_1.$$

Ratios of the Gauss functions

Küstner (2002): if $-1 < a_2 \le b_1$ and $0 < a_1 \le b_1$ then

$$f_2(z) = \frac{{}_2F_1(a_1, a_2+1; b_1; z)}{{}_2F_1(a_1, a_2; b_1; z)} \in \mathcal{T} \quad \text{and}$$

$$f_3(z) = \frac{{}_2F_1(a_1+1, a_2+1; b_1+1; z)}{{}_2F_1(a_1, a_2; b_1; z)} \in \mathcal{T}$$

K.-Dyachenko (work in progress). Under certain conditions on parameters:

$$f_2(z) = B_1 \int_0^1 \frac{t^{a_1+a_2-1}(1-t)^{b_1-a_1-a_2-1}dt}{(1-zt)|{}_2F_1(a_1, a_2; b_1; 1/t)|^2},$$

$$f_3(z) = B_2 \int_0^1 \frac{t^{a_1+a_2}(1-t)^{b_1-a_1-a_2-1}dt}{(1-zt)|{}_2F_1(a_1, a_2; b_1; 1/t)|^2},$$

where $B_1 = [\Gamma(b_1)]^2/\Gamma(a_1)\Gamma(a_2+1)\Gamma(b_1-a_1)\Gamma(b_1-a_2)$,

$B_2 = b_1[\Gamma(b_1)]^2/\Gamma(a_1+1)\Gamma(a_2+1)\Gamma(b_1-a_1)\Gamma(b_1-a_2)$.

Ratios of the GHF

K.-Dyachenko (work in progress). Suppose $\alpha \geq 0$, $\mathbf{a}, \mathbf{b} > 0$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ for $t \in (0,1)$, then

$$F(z) = {}_{p+1}F_p\left(\begin{matrix} 1, \mathbf{a}+\alpha \\ \mathbf{b}+\alpha \end{matrix} \middle| z\right) \Big/ {}_{p+1}F_p\left(\begin{matrix} 1, \mathbf{a} \\ \mathbf{b} \end{matrix} \middle| z\right) \in \mathcal{T}.$$

The density $\mu(t)$ of representing measure is given by:

$$\frac{\Gamma(\mathbf{a})\Gamma(\mathbf{a}+\alpha)}{\pi\Gamma(\mathbf{b})\Gamma(\mathbf{b}+\alpha)}|{}_{p+1}F_p\left(1,\mathbf{a};\mathbf{b};1/t\right)|^2 t\mu(t) = G^{p,0}_{p,p}\left(t\middle| \begin{matrix} \mathbf{b} \\ \mathbf{a} \end{matrix}\right) \times$$
$$\left\{G^{p+1,1}_{p+2,p+2}\left(t\middle| \begin{matrix} 1, 3/2, \mathbf{b}+\alpha \\ 1, \mathbf{a}+\alpha, 3/2 \end{matrix}\right) - t^{\alpha} G^{p+1,1}_{p+2,p+2}\left(t\middle| \begin{matrix} 1, 3/2, \mathbf{b} \\ 1, \mathbf{a}, 3/2 \end{matrix}\right)\right\}.$$

Nevanlinna classes

Nevanlinna class $\mathcal{N}_\kappa$

A function $\varphi(z)$ belongs to $\mathcal{N}_\kappa$ whenever it is meromorphic in $\Im z > 0$, and for any set of non-real points $z_1, \ldots, z_k$ the Hermitian form

$$H_\varphi(\zeta_1, \ldots, \zeta_k | z_1, \ldots, z_k) = \sum_{n,m=0}^{k} \frac{\varphi(z_n) - \overline{\varphi(z_m)}}{z_n - \overline{z_m}} \zeta_n \overline{\zeta_m}$$

has at most κ negative squares and for some set of points exactly κ negative squares.

The class $\mathcal{N}_0$ coincides with Nevanlinna-Pick class of holomorphic functions mapping the upper half-plane into itself.

Conjecture (partially proved for ratios of ${}_2F_1(z)$)

For all real parameters the ratios $f_1(z)$, $f_2(z)$, $f_3(z)$ and $F(z)$ belong to $\mathcal{N}_\kappa$ with κ explicitly expressed in terms of parameters.

Analytic continuation in parameters

Shorthand notation: ${}_{p+1}F_p\left(\mathbf{a};\mathbf{b}\right) := {}_{p+1}F_p\left(\mathbf{a};\mathbf{b};1\right)$

${}_2F_1(1)$ - Gauss (1812) formula (the series on the left converges for $\Re(b_1 - a_1 - a_2) > 0$):

$${}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}\right) = \frac{\Gamma(b_1)\Gamma(b_1 - a_1 - a_2)}{\Gamma(b_1 - a_1)\Gamma(b_1 - a_2)}$$

Note that hyper-planes $b_1 - a_1 - a_2 \in -\mathbb{N}_0$ are poles.

${}_3F_2(1)$ - Kummer (1836) and Thomae (1879) relations ($\psi = b_1 + b_2 - a_1 - a_2 - a_3$):

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right) = \frac{\Gamma(b_2)\Gamma(\psi)}{\Gamma(b_2 - a_1)\Gamma(\psi + a_1)}{}_3F_2\left(\begin{matrix} a_1, b_1 - a_2, b_1 - a_3 \\ b_1, \psi + a_1 \end{matrix}\right),$$

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right) = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(\psi)}{\Gamma(a_1)\Gamma(\psi + a_2)\Gamma(\psi + a_3)}{}_3F_2\left(\begin{matrix} b_1 - a_1, b_2 - a_1, \psi \\ \psi + a_2, \psi + a_3 \end{matrix}\right).$$

Analytic continuation in parameters

General case ${}_{p+1}F_p$ - Olsson (1966), Bühring (1992), K.-Prilepkina (2018): for $\Re(\psi) > 0$ and $\Re(\mathbf{a}_{[1,2]}) > 0$:

$$
{}_{p+1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) = \frac{\Gamma(\mathbf{b})\Gamma(\psi)}{\Gamma(\mathbf{a}_{[1,2]})\Gamma(\psi+a_1)\Gamma(\psi+a_2)}\sum_{n=0}^{\infty}\frac{(\psi)_n g_n(\mathbf{a}_{[1,2]};\mathbf{b})}{(\psi+a_1)_n(\psi+a_2)_n},
$$

where $g_n(\boldsymbol{\alpha};\boldsymbol{\beta})$ are Nørlund's coefficients defined either by recurrence relations or by $p-2$-fold summation (finally $p-1$-fold summation).

Theorem (K.-Prilepkina, 2018) - triple summation

Recall that $\psi = \sum_{k=1}^{p} b_k - \sum_{j=1}^{p+1} a_j$. For $\Re(\psi) > 0$ and $\Re(\mathbf{a}_{[1,2]}) > 0$:

$$
\frac{\Gamma(\mathbf{a}_{[1,2]})}{\Gamma(\mathbf{b})}\,{}_{p+1}F_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\right) = \frac{\pi}{\sin(\pi\psi)}\sum_{k=1}^{p}\frac{\Gamma(b_k-\mathbf{b}_{[k]})}{\Gamma(b_k-\mathbf{a})}\times
$$

$$
\sum_{n=0}^{\infty}\frac{(1-b_k+a_1)_n(1-b_k+a_2)_n}{\Gamma(1-b_k+a_1+a_2+n)n!}\,{}_pF_{p-1}\left(\begin{matrix}-n, 1-b_k+\mathbf{a}_{[1,2]}\\ 1-b_k+\mathbf{b}_{[k]}\end{matrix}\right).
$$

Theorem (K.-López, 2016)

Suppose $0 < \alpha \leq 1$, $\beta_1 \geq \alpha + 1$, $\beta_2 \geq 3/2$, $\mathbf{a} > 0$ and $v_{\mathbf{a},\mathbf{b}}(t) \geq 0$ on $[0, 1]$. Then

$$0 < {}_{p-1}F_p(\alpha, \mathbf{a}; \beta_1, \beta_2, \mathbf{b}; x) < 1$$

for all $x < 0$. In particular, this function has no real zeros.

Theorem (K.-López, 2016)

Let $\mathbf{a}$, $\mathbf{b}$ be positive vectors. Suppose that $a_k \leq \min\{1, b_s - 1\}$ for some indexes $k, s \in \{1, \ldots, p\}$ and $v_{\mathbf{a}_{[k]},\mathbf{b}_{[s]}}(t) \geq 0$ on $[0, 1]$. Then ${}_pF_p(\mathbf{a}; \mathbf{b}; z)$ has no real zeros and all its zeros lie in the open right half plane $\Re(z) > 0$. Here $\mathbf{a}_{[k]} = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_p)$.

Laguerre-Pólya class

Laguerre-Pólya class $\mathcal{L}-\mathcal{P}$: real entire functions with factorization

$$f(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

where $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, $n \in \mathbb{N}_0$ and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$.

Theorem C (Richards, 1989; Ki and Kim, 2000)

Suppose $p \leq q$, $\mathbf{a}, \mathbf{b} > 0$ and $\mathbf{a}$ can be re-indexed so that $a_k = b_k + n_k$ for $n_k \in \mathbb{N}_0$ and $k = 1, \ldots, p$. Then

$$\phi(z) = {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = e^{az} \prod_{k=1}^{\omega} \left(1 + \frac{z}{z_k}\right) e^{-\frac{z}{z_k}} \in \mathcal{L}-\mathcal{P},$$

where $z_k > 0$, $\omega \leq \infty$ and the series $\sum_{n=1}^{\infty} 1/z_n^2$ converges. Furthermore, if $p = q$, $\mathbf{a} \in \mathbb{R}$ contains no non-positive integers and $\mathbf{b} > 0$ then $a_k = b_k + n_k$ for $n_k \in \mathbb{N}_0$ is necessary and sufficient for $\phi \in \mathcal{L}-\mathcal{P}$.

Extended Laguerre inequalities

Theorem D (Patrick, 1973; Csordas and Varga, 1990)

$$\text{Let } f(z) = e^{-bz^2} f_1(z), \quad (b \geq 0, f(z) \not\equiv 0),$$

where $f_1(z)$ is a real entire function of genus 0 or 1. Set

$$L_n[f](x) = \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x)$$

for $x \in \mathbb{R}$ and $n \geq 0$. Then $f(z) \in \mathcal{L}-\mathcal{P}$ if and only if

$$L_n[f](x) \geq 0 \text{ for all } x \in \mathbb{R} \text{ and } n \geq 0.$$

Corollary: extended Laguerre inequalities

Under hypotheses of Theorem C $L_n[{}_pF_q(\mathbf{a};\mathbf{b};x)] \geq 0$ for all integer $n \geq 0$ and all $x \in \mathbb{R}$.

Laguerre inequalities

Theorem (Kalmykov-K., 2017): Laguerre inequality

If $p \le q$ and conditions $\mathbf{a} \prec^W \mathbf{b}'$ are satisfied, where $\mathbf{b}'$ stands $\mathbf{b}$ with $q-p$ largest elements removed. Then the function $x \to {}_pF_q(x)$ satisfies the Laguerre inequality

$$ {}_pF_p'\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right)^2 - {}_pF_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) {}_pF_p''\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) \ge 0. $$

Conjecture: zeros of ${}_pF_q$

Suppose $p < q$, $\mathbf{b} > 0$ and $a_k > b_k$ for $k = 1, \ldots, p$. Then all zeros of ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$ are real and negative.

Craven and Csordas (2006) conjectured that the following function has only real and negative zeros for each positive integer m

$$ {}_{m-1}F_m\left(\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}; \frac{1}{m+1}, \frac{2}{m+1}, \ldots, \frac{m}{m+1}; z\right) $$

THANK YOU FOR ATTENTION!