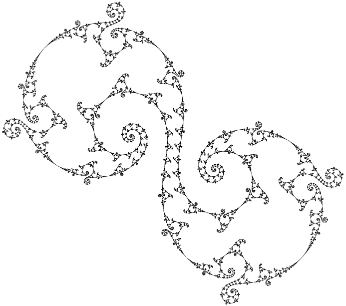
Semigroups of hyperbolic isometries



CAFT 2018

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- When $\mathbb{B}$ is equipped with the hyperbolic metric ρ, the group $\mathcal{M}_3$ is exactly the group of orientation preserving isometries of $(\mathbb{B}, \rho)$, and $\widehat{\mathbb{C}}$ is its ideal boundary.

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- When $\mathbb{B}$ is equipped with the hyperbolic metric ρ, the group $\mathcal{M}_3$ is exactly the group of orientation preserving isometries of $(\mathbb{B}, \rho)$, and $\widehat{\mathbb{C}}$ is its ideal boundary.
- We shall also consider the subgroup $\mathcal{M}_2 \subset \mathcal{M}_3$ that fixes $\mathbb{D}$ set-wise and preserves orientation on $\mathbb{D}$.

Möbius semigroups

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Throughout we shall restrict our attention to *finitely-generated* Möbius semigroups, and refer to these simply as semigroups.

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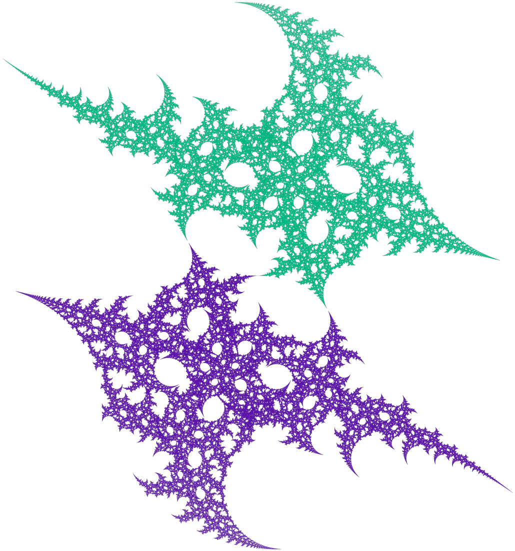
Examples: Kleinian groups, $S = \left\langle z \mapsto \frac{1}{3}z,\ z \mapsto \frac{1}{3}z + \frac{2}{3} \right\rangle$.



$\Lambda^+(S)$ and $\Lambda^-(S)$ where $S = \left\langle z \mapsto \frac{a}{1+z},\ z \mapsto \frac{a-1+2ia^{1/2}}{1+z},\ z \mapsto \frac{1}{4(1+z)} \right\rangle$, $a = -0.1 + 0.7i$.

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Suppose that $S \subseteq \mathcal{M}_3$ *is a nonelementary, semidiscrete semigroup. If* $\Lambda^+(S) = \Lambda^-(S)$ *and this set is not connected, then* S *is a group.*

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Conjecture

Suppose that $S \subseteq \mathcal{M}_3$ is a nonelementary semidiscrete semigroup. If $\Lambda^+(S) = \Lambda^-(S) \neq \widehat{\mathbb{C}}$, then S is a group.

If the forward and backward limit sets are equal, then the following Lemma tells us the semigroup is contained in a Kleinian group.

Lemma

Suppose S is a nonelementary semidiscrete semigroup, and that $\Lambda^+(S) = \Lambda^-(S) = \Lambda$, where Λ is not a circle nor $\widehat{\mathbb{C}}$. Then the elements of $\mathcal{M}_3$ that fix Λ setwise form a discrete group.

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  Again, we denote the conjugate of G and S by ϕ as Γ and Σ respectively.

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  Then Γ is a Fuchsian group of the first kind.

The case where Λ is a decomposable continuum

A continuum Λ is *decomposable* if there exist subcontinua λ_1 and λ_2, neither empty nor Λ itself, such that $\Lambda = \lambda_1 \cup \lambda_2$.
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Theorem (Matsuzaki 2004)

Let G be a Kleinian group such that $\Omega(G)$ has one component and $\Lambda(G)$ is a decomposable continuum. Let $\phi : \mathbb{D} \to \Omega(G)$ be a Riemann map, and suppose $\phi^{-1} G \phi = \Gamma$ is a Fuchsian group of the 1st kind.
If $E \subseteq \mathbb{S}^1$ is not dense in $\mathbb{S}^1$ then $I(E) \subsetneq \Lambda$.

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Proposition (J. 2018)

Let S be a semidiscrete semigroup such that $\Lambda^+(S) = \Lambda^-(S)$ is a decomposable continuum whose complement has one component. Then S is a group.

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By passing to a subsequence if necessary, $\phi^{-1} f_n \phi$ converges ideally to some point $w \in \Lambda^+(\Sigma)$.
Since $z \in I(\overline{\phi}(w)) \subseteq I(\Lambda^+(\Sigma))$ we have a contradiction. □

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Bishop–Jones $H^{\frac{1}{2}-\eta}$ Theorem (Bishop, Jones 1994)

Let ϕ be a conformal mapping from $\mathbb{D}$ onto Ω. If

$$\iint_{\mathbb{D}} |\phi'(z)||\mathcal{S}(\phi)(z)|^2(1-|z|^2)^3 \, dxdy < +\infty,$$

then ϕ' belongs to the Hardy space $H^{\frac{1}{2}-\eta}$ for all $\eta \in (0, 1/2)$.

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Theorem (Bishop, Jones 1994)

If a finitely-generated Kleinian group has a simply connected invariant component that is not a disc, then a.e. point on the boundary with respect to harmonic measure is a twist point.

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The Schwarzian of ϕ is Γ-invariant because $\mathcal{S}(\phi\gamma) = \mathcal{S}(g\phi) = \mathcal{S}(\phi)$ whenever $g \in G$, $\gamma \in \Gamma$ and $g = \phi\gamma\phi^{-1}$.

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Recall the Bishop–Jones integral

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Conjecture

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Proposition (J. 2018)

Suppose that $S \subseteq \mathcal{M}_3$ is a nonelementary semidiscrete semigroup. If $\Lambda^+(S) = \Lambda^-(S) \neq \widehat{\mathbb{C}}$ and is not a connected set with infinitely many complementary components, then S is a group.

Thank you for your attention!