Describing Blaschke products by their critical points

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Finite Blaschke Products

A finite Blaschke product of degree $d \geq 1$ is an analytic function from $\mathbb{D} \to \mathbb{D}$ of the form

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[Here, unique = unique up to post-composition with a Möbius transformation in $\mathrm{Aut}(\mathbb{D})$.]

Inner functions

An inner function is a holomorphic self-map of $\mathbb{D}$ such that for almost every $\theta \in [0, 2\pi)$, the radial limit

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Different inner functions can have the same critical set. For example, $F_1(z) = z$ and $F_2(z) = \exp(\frac{z+1}{z-1})$ have no critical points.

BS decomposition

An inner function can be represented as a (possibly infinite) Blaschke product $\times$ singular inner function:

$$B = e^{i\psi} \prod_i -\frac{\overline{a_i}}{|a_i|} \cdot \frac{z - a_i}{1 - \overline{a_i} z}, \quad a_i \in \mathbb{D}, \quad \sum (1 - |a_i|) < \infty.$$

$$S = \exp\left(- \int_{\mathbb{S}^1} \frac{\zeta + z}{\zeta - z} d\sigma_\zeta \right), \quad \sigma \perp m, \quad \sigma \geq 0.$$

Here, B records the zero set, while S records the boundary zero structure.

Inner functions of finite entropy

We will also be concerned with the subclass $\mathscr{J}$ of inner functions whose derivative lies in the Nevanlinna class:

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F'(re^{i\theta})| d\theta < \infty.$$

In 1974, P. Ahern and D. Clark showed that F' admits a BSO decomposition, allowing us to define $\text{Inn}\, F' := BS$, where B records the critical set of F and S records the boundary critical structure.

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is injective but NOT surjective. The image consists of all inner functions of the form BS_μ where B is a Blaschke product and μ is a measure supported on a countable union of Beurling-Carleson sets.

Beurling-Carleson sets

Definition. A Beurling-Carleson set E is a closed subset of the unit circle which has measure 0 such that

$$\sum |I_j| \cdot \log \frac{1}{|I_j|} < \infty,$$

where $\{I_j\}$ are the complementary intervals.

[Measures which do not charge Beurling-Carleson sets also occur in the description of cyclic functions in Bergman spaces given indepedently by Korenblum (1977) and Roberts (1979).]

Background on conformal metrics

The curvature of a conformal metric $\lambda(z)|dz|$ is given by

$$k_\lambda = -\frac{\Delta \log \lambda}{\lambda^2}.$$

Examples. The hyperbolic metric

$$\lambda_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$$

has curvature $\equiv -1$,

while the Euclidean metric $|dz|$ has curvature $\equiv 0$.

Liouvillean correspondence

Since curvature is a conformal invariant, if $F : \mathbb{D} \to \mathbb{D}$ is a holomorphic map then

$$\lambda_F \;=\; F^*\lambda_{\mathbb{D}} \;=\; \frac{2|F'|}{1-|F|^2}$$

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Liouville observed that there is a natural bijection between $\mathrm{Hol}(\mathbb{D}, \mathbb{D}) / \operatorname{Aut} \mathbb{D}$ and pseudometrics of constant curvature -1 with integral singularities.

Nearly-maximal solutions

Consider the Gauss curvature equation

$$\Delta u = e^{2u}, \qquad u : \mathbb{D} \to \mathbb{R}.$$

It has a unique maximal solution $u_{\max} = \log \lambda_{\mathbb{D}}$ which tends to infinity as $|z| \to 1$.

We are interested in solutions close to maximal in the sense that

$$\limsup_{r \to 1} \int_{|z|=r} (u_{\max} - u) d\theta < \infty.$$

Embedding into the space of measures

For each $0 < r < 1$, we may view

$$(u_{\max} - u)d\theta$$

as a positive measure on the circle of radius r.

Subharmonicity guarantees the existence of a weak limit as $r \to 1$, which we denote $\mu[u]$.

It turns out that the measure μ uniquely determines the solution u. Thus, the question becomes: which measures occur?

Constructible measures

Theorem. (I, 2017) Any measure μ on the unit circle can be uniquely decomposed into a constructible part and an invisible part:

$$\mu = \mu_{\text{con}} + \mu_{\text{inv}}.$$

In fact, $u_{\mu_{\text{con}}}$ is the **minimal solution** which exceeds the subsolution $u_{\max} - P_\mu$ (Poisson extension).

Remark. The above theorem holds for other PDEs such as $\Delta u = |u|^{q-1}u$, $q > 1$, any smooth bounded domain, and is valid in higher dimensions.

Cullen's Theorem

Theorem. (M. Cullen, 1971) If a measure ν is supported on a Beurling-Carleson set, then $S'_\nu \in \mathcal{N}$.

In particular,

$$u = \log \frac{2|S'_\nu|}{1 - |S_\nu|^2} \quad \text{is} \quad \textbf{nearly-maximal},$$

i.e. ν is constructible.

From my work, it follows that Cullen's theorem is essentially sharp: if $S'_\mu \in \mathcal{N}$, then μ lives on a countable union of Beurling-Carleson sets. Artur Nicolau gave an elementary proof of this fact.

Roberts' decompositions

Claim. If $\omega_\mu(t) \leq c \cdot t \log(1/t)$, then μ is invisible.

[The modulus of continuity $\omega_\mu(t) = \sup_{I \subset \mathbb{S}^1} \mu(I)$, with the supremum taken over all intervals of length t.]

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Theorem. (J. Roberts, 1979) Suppose μ does not charge Beurling-Carleson sets. Given a real number $c > 0$ and integer $j_0 \ge 1$, μ can be expressed as a countable sum

$$\mu = \sum_{j=1}^{\infty} \mu_j,$$

where

$$\omega_{\mu_j}(1/n_j) \le \frac{c}{n_j} \cdot \log n_j, \qquad n_j := 2^{2^{j+j_0}}.$$

On L^1 bounded solutions

Consider the differential equation

$$\Delta u = |u|^{q-1}u, \qquad u : \mathbb{B} \to \mathbb{R}, \quad q > 1,$$

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^N$. We say that u is an L^1 bounded solution if

$$\limsup_{r \to 1} \int_{\mathbb{B}} |u(r\xi)| d\sigma < \infty.$$

Taking the weak limit of $u(r\xi)\, d\sigma$ as $r \to 1$, one obtains an embedding of L^1 bounded solutions into $\mathcal{M}(\partial\mathbb{B})$.

Question. Which measures occur (are constructible)?

On L^1 bounded solutions

Theorem. (A. Gmira & L. Véron, 1991) In the **subcritical case**, $q < q_c = \frac{N+1}{N-1}$, all measures are constructible.

Theorem. In the **supercritical case**, $q \geq q_c$, a measure is constructible iff it is diffuse with respect to $\mathrm{cap}_{W^{2/q,q'}}$.

This was proved by:

- J. F. Le Gall, $q = 2$ (1993),
- E. B. Dynkin & S. E. Kuznestov, $q_c \leq q \leq 2$ (1996),
- M. Marcus & L. Véron, $q > 2$ (1998).

Stable topology on inner functions

Endow $\mathscr{J} \,/\, \mathrm{Aut}\,\mathbb{D}$ with the stable topology where $F_n \to F$ if

- The convergence is uniform on compact subsets of the disk,
- The Nevanlinna splitting is stable in the limit:

$$\mathrm{Inn}\, F_n' \to \mathrm{Inn}\, F', \qquad \mathrm{Out}\, F_n' \to \mathrm{Out}\, F'.$$

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Theorem. (I, 2018) This happens if and only if the "critical structures" of the F_n are uniformly concentrated on Korenblum stars.

Critical structures of inner functions

Consider the weighted Bergman space $A^2_1(\mathbb{D})$ which consists of all holomorphic functions on the unit disk satisfying the norm boundedness condition

$$\|f\|_{A^2_1} = \left(\int_{\mathbb{D}} |f(z)|^2 \cdot (1 - |z|)|dz|^2 \right)^{1/2} < \infty.$$

Theorem. (D. Kraus, 2007) Critical sets of inner functions = Zero sets of the weighted Bergman space A^2_1.

It therefore makes sense to seek a bijection between $\mathrm{Inn} / \mathrm{Aut}\, \mathbb{D}$ and certain invariant subspaces of A^2_1.

Invariant subspaces of Bergman spaces

Conjecture. $\mathrm{Inn} \,/\, \mathrm{Aut}\,\mathbb{D} \cong \overline{\{\text{zero-based subspaces}\}}$.

A subspace is zero-based if consists of functions which vanish on a prescribed set of points.

We say that $X_n \to X$ if any $x \in X$ can be obtained as a limit of a converging sequence of $x_n \in X_n$ and visa versa.

Theorem. (I, 2018) The collection of z-invariant subspaces of A^2_1 which are generated by a single inner function is naturally homeomorphic to $\mathscr{J} \,/\, \mathrm{Aut}\,\mathbb{D}$.

Thank you for your attention!