The numerical range and compressions of the shift operator

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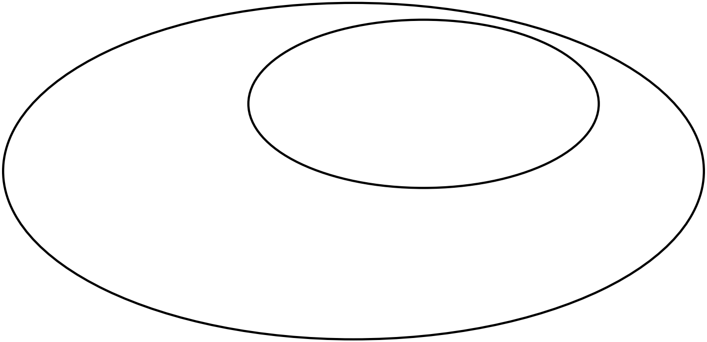
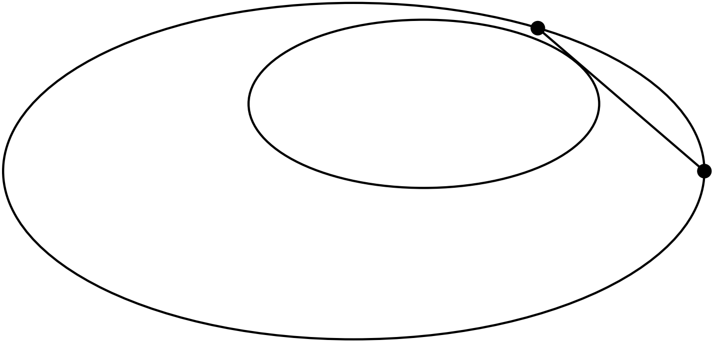
July 2018

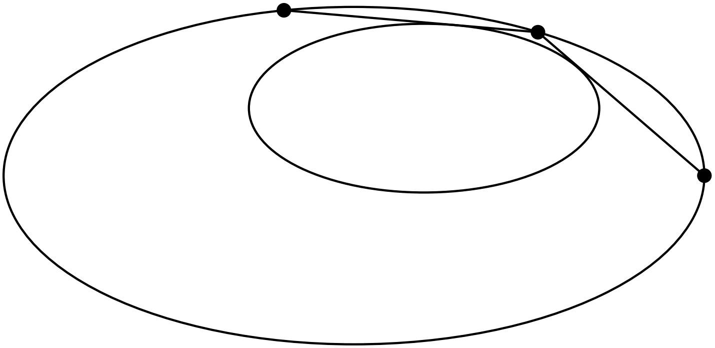
Poncelet and projective geometry

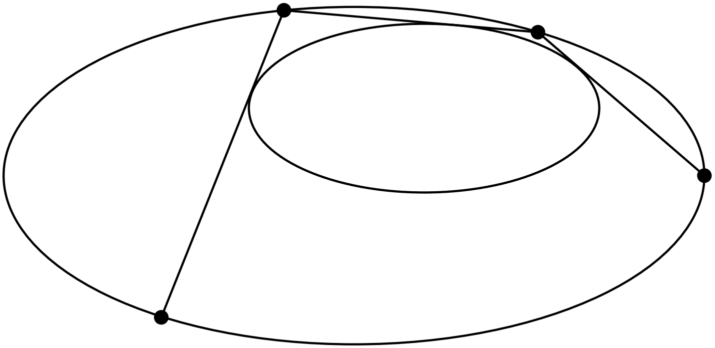
Theorem

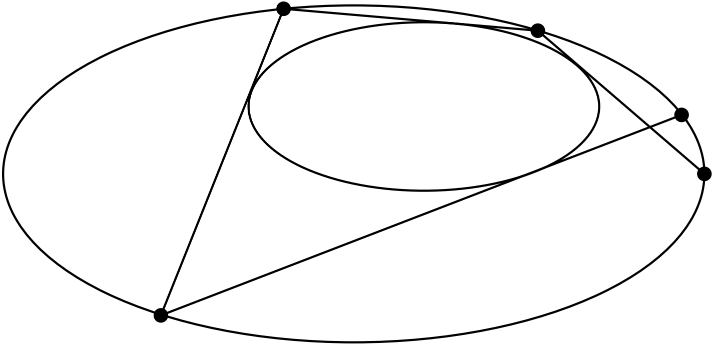
(Poncelet's Porism, 1813, ellipse version) Given one ellipse inside another, if there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the inner) n -gon, then any point on the boundary of the outer ellipse is the vertex of some circuminscribed n-gon.

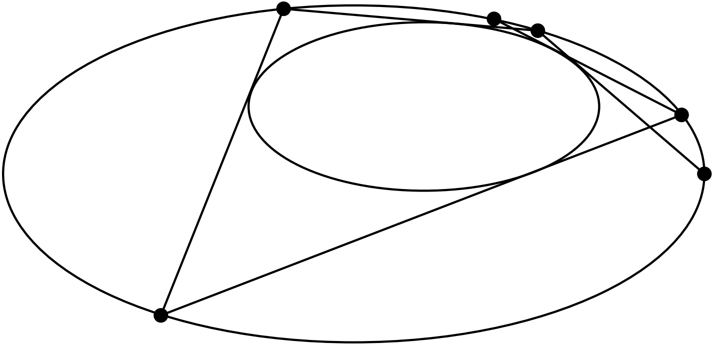


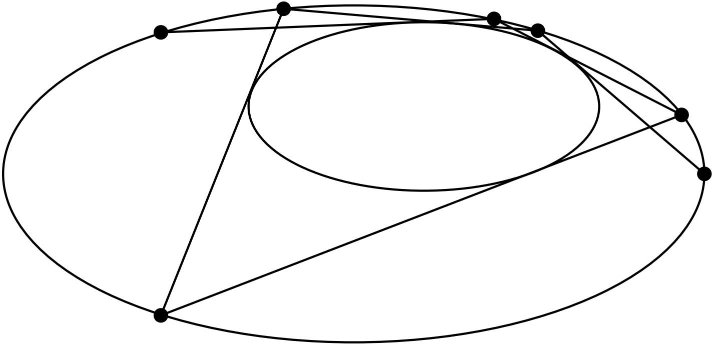


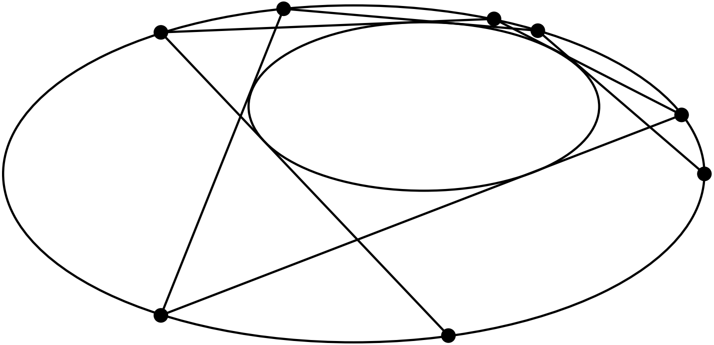


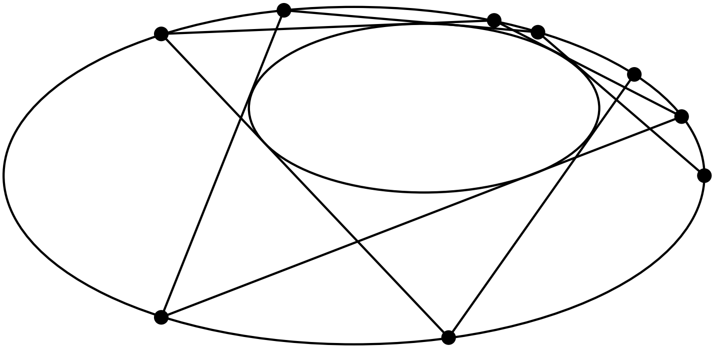




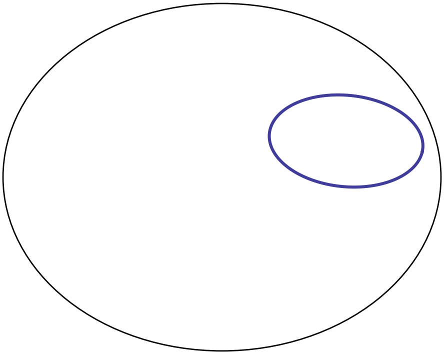


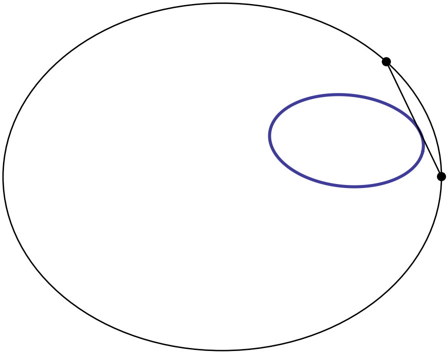


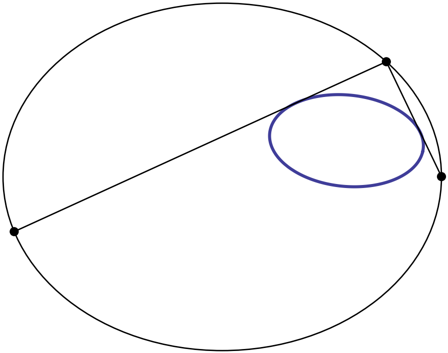


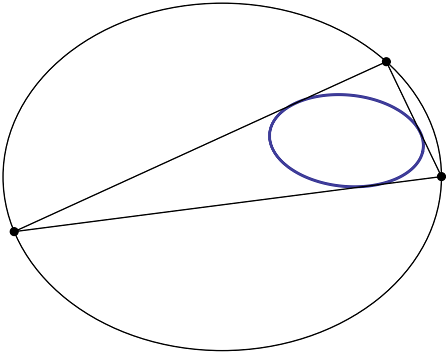


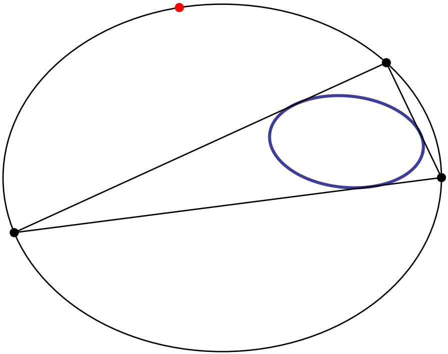
Maybe we never returning to the starting point. Maybe, though, we do return to the initial point.

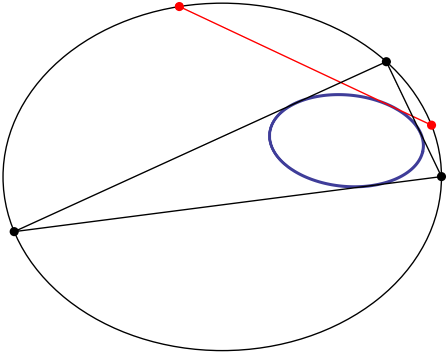


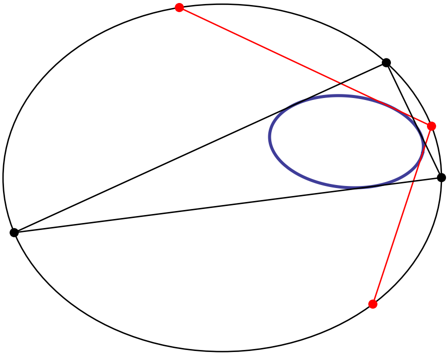


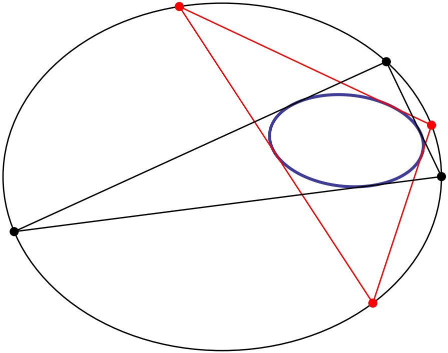


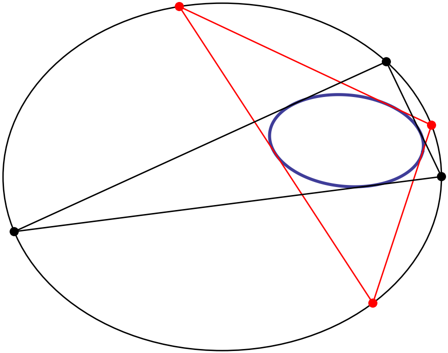




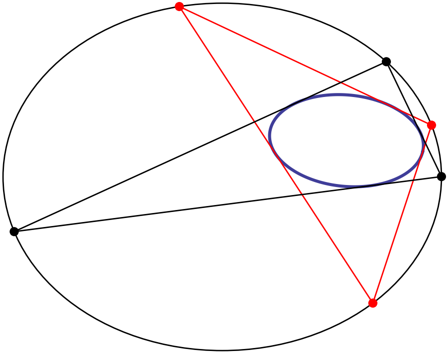








Poncelet's theorem says that if the path closes in n steps, then *no matter where you begin* the path will close in n steps.



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New proof Halbeisen and Hungerbühler, 2015!

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Leopold Flatto, *Poncelet's Theorem*, dynamics perspective

Hold that thought

Numerical range

A an $n \times n$ matrix.

The *numerical range* of A is $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$.

Why the numerical range?

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Compare the zero matrix and the $n \times n$ Jordan block: (Here's the 2×2)

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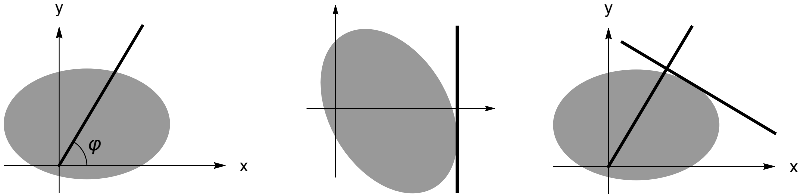
$$W(A_1) = \{0\}, W(A_2) = \{z : |z| \leq 1/2\}.$$

Kippenhahn's work

Kippenhahn: Finding the numerical range

Idea: Find the maximum eigenvalue of $(A + A^{\star})/2$. Then rotate A and repeat.



Theory of envelopes and projective geometry

The Envelope Algorithm

Have a family of curves $\mathcal{F}$ given by $F(x, y, \theta) = 0$.

Find $F_\theta(x, y, \theta) = 0$.

Solve for one variable.

Get the equation of a curve each point of which is a point of tangency to some member of $F(x, y, \theta)$.

The envelope three ways and the boundary

1. Find a curve $\mathcal{C}$ such that every point of $\mathcal{C}$ is tangent to a member of $\mathcal{F}$ (and sometimes every member of the family is tangent to the curve).

2. Find a curve satisfying the envelope algorithm.

3. For each θ choose two curves C_θ and $C_{\theta+h}$ and find the points of intersection. The envelope consists of the points obtained from
$$\lim_{h \to 0} C_\theta \cap C_{\theta+h}.$$

These are not always the same, but for us they will be.

Numerical range basics

Elliptical range theorem

Theorem

Let A be a 2×2 matrix with eigenvalues a and b. Then the numerical range of A is an elliptical disk with foci at a and b and minor axis given by $(tr(A^\star A) - |a|^2 - |b|^2)^{1/2}$.

Why?

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$x = \begin{bmatrix} te^{i\theta_1} \\ \sqrt{1-t^2}e^{i\theta_2} \end{bmatrix}$. Then

$$\langle Ax, x\rangle = (1-t^2) + me^{i(\theta_2-\theta_1)}(t\sqrt{1-t^2}).$$

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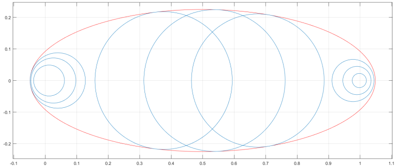

We now find the envelope of the family of circles.

We had

$$F(x, y, t) := (x - (1 - t^2))^2 + y^2 - m^2 t^2 (1 - t^2) = 0.$$

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Computing $F_t(x, y, t) = 0$ when

$$x = (1 - t^2) + \frac{m^2}{2}(1 - 2t^2) \text{ and } y^2 = m^2(t^2 - t^4) - \frac{m^4}{4}(1 - 2t^2)^2.$$

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Combining the formulas for x and y shows that

$$\frac{(x - \frac{1}{2})^2}{1 + m^2} + \frac{y^2}{m^2} = \frac{1}{4}. \tag{1}$$

Is the envelope the boundary?

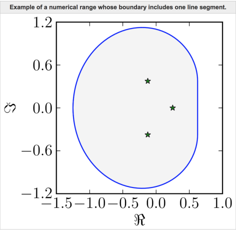
(Details Trung Tran, Kelly Bickel + G.)

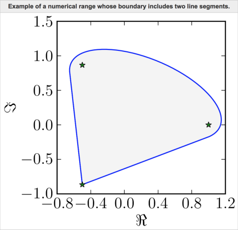
An important consequence of the elliptical range theorem

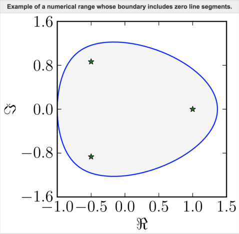
Theorem (The Toeplitz-Hausdorff Theorem; 1918)

The numerical range of an $n \times n$ matrix is convex.

Some possible shapes







Source:`http://numericalshadow.org/doku.php?id=numerical-range:examples:3x3`

Remark: Every unitary matrix is unitarily equivalent to a diagonal matrix, with its eigenvalues on the diagonal. If

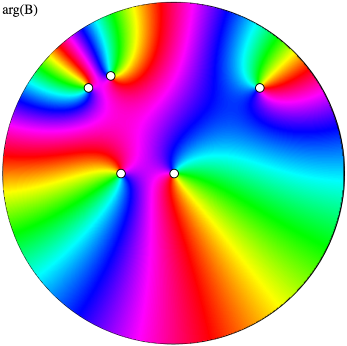
$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

then $\langle A_1 x, x \rangle = \sum_{j=1}^{3} \lambda_j |x_j|^2$, which is the convex hull of the eigenvalues.

Fact: The numerical range of a unitary matrix is the convex hull of its eigenvalues.

The numerical range of a compressed shift operator

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \text{ where } a_j \in \mathbb{D}, |\lambda| = 1.$$



Visualizing Blaschke products

H^2 is the Hardy space; $f(z) = \sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

An inner function is a bounded analytic function on $\mathbb{D}$ with radial limits of modulus one almost everywhere.

S is the shift operator $S : H^2 \to H^2$ defined by $[S(f)](z) = zf(z)$;

The adjoint is $[S^\star(f)](z) = (f(z) - f(0))/z$.

Theorem (Beurling's theorem)

The nontrivial invariant subspaces under S are

$$UH^2 = \{Uh : h \in H^2\},$$

where U is a (nonconstant) inner function.

Subspaces invariant under the adjoint, $S^\star$ are $K_U := H^2 \ominus UH^2$.

What's the model space?

Theorem

Let U be inner. Then $K_U = H^2 \cap U\,\overline{zH^2}$.

So $\{f \in H^2 : f = U\overline{gz}\; a.e.$ for some $g \in H^2\}$.

Consider $K_B = H^2 \ominus BH^2$ where $B(z) = \prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a_j}z}$

and the Szegö kernel: $g_a(z) = \dfrac{1}{1-\overline{a}z}$.

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- So $\langle Bh, g_{a_j} \rangle = B(a_j)h(a_j) = 0$ for all $h \in H^2$.

So $g_{a_j} \in K_B$ for $j = 1, 2, \ldots, n$.

If a_j are distinct, $K_B = \text{span}\{g_{a_j} : j = 1, \ldots, n\}$.

Compressions of the shift

Consider the compression of the shift: $S_B : K_B \to K_B$ defined by

$$S_B(f) = P_B(S(f))$$

where P_B is the orthogonal projection from H^2 onto K_B.

Applying Gram-Schmidt to the kernels we get the Takenaka-Malmquist basis: Let $b_a(z) = \frac{z-a}{1-\overline{a}z}$ and

$$\{\frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, b_{a_1}\frac{\sqrt{1-|a_2|^2}}{1-\overline{a_2}z}, \ldots \prod_{j=1}^{k-1} b_{a_j}\frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}z}, \ldots\}.$$

What's the matrix representation for S_B with respect to this basis?

For two zeros it's

$$A = \begin{bmatrix} a & \sqrt{1-|a|^2}\sqrt{1-|b|^2} \\ 0 & b \end{bmatrix}.$$

So A is the matrix representing S_B when B has two zeros a and b. The numerical range is an elliptical disk with foci at a and b.

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What about the $n \times n$ case?

The $n \times n$ matrix A is

$$\begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & \dots & \left(\prod_{k=2}^{n-1}(-\overline{a_k})\right)\sqrt{1-|a_1|^2}\sqrt{1-|a_n|^2} \\ 0 & a_2 & \dots & \left(\prod_{k=3}^{n-1}(-\overline{a_k})\right)\sqrt{1-|a_2|^2}\sqrt{1-|a_n|^2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_n \end{bmatrix}$$

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For each $\lambda \in \mathbb{T}$, we have A "inside" a unitary matrix

$$b_{ij} = \begin{cases} a_{ij} & \text{if } 1 \leq i, j \leq n, \\ \lambda\left(\prod_{k=1}^{j-1}(-\overline{a_k})\right)\sqrt{1-|a_j|^2} & \text{if } i = n+1 \text{ and } 1 \leq j \leq n, \\ \left(\prod_{k=i+1}^{n}(-\overline{a_k})\right)\sqrt{1-|a_i|^2} & \text{if } j = n+1 \text{ and } 1 \leq i \leq n, \\ \lambda \prod_{k=1}^{n}(-\overline{a_k}) & \text{if } i = j = n+1. \end{cases}$$

Examples

1. Let $B(z) = z^n$. Then
$$K_B = \text{ span}(1, z, z^2, \ldots, z^{n-1})$$
2. S_B can be represented by
$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Halmos and unitary dilations

Let $\|A\| \leq 1$. Look at $S = \sqrt{1 - AA^\star}$ and $T = \sqrt{1 - A^\star A}$.

Then

$$U = \begin{pmatrix} A & S \\ T & -A^\star \end{pmatrix}$$

is a unitary dilation of A.

Halmos asked: What do the unitary dilations tell us about A? Specifically, is

$$\overline{W(A)} = \bigcap \{\overline{W(U)} : U \text{ a unitary dilation of } A\}?$$

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Let $V = [I_n, 0]$ be $n \times (n+1)$. Then $A = VU_\lambda V^t$ and $V^t x = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $\|V^t x\| = 1$.

For compressions of the shift

$$U_\lambda = \begin{bmatrix} A & \text{stuff}(\lambda) \\ \text{stuff}(\lambda) & \text{stuff}(\lambda) \end{bmatrix} \leftarrow \text{ add one row and one column}$$

$$\text{rank}(I - S_B^\star S_B) = 1 = \text{rank}(I - S_B S_B^\star)$$

1. The eigenvalues of U_λ are the values $\hat{B}(z) := zB(z)$ maps to λ;
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$$\langle Ax, x\rangle = \langle VU_\lambda V^t x, x\rangle = \langle U_\lambda V^t x, V^t x\rangle.$$

Our operators

$\mathcal{S}_n$ denotes compressions of the shift to an n-dimensional space:

Matrices have no eigenvalues of modulus 1, are contractions (completely non-unitary contractions) with $\text{rank}(I - T^\star T) = \text{rank}(I - TT^\star) = 1$.

B be a finite Blaschke product, $K_B = H^2 \ominus BH^2 = H^2 \cap B\overline{zH^2}$.

$$S_B(f) = P_B(S(f)) \text{ where } f \in K_B, P_B : H^2 \to K_B.$$

$$P_B(g) = BP_-(\overline{B}g) = B(I - P_+)(\overline{B}g),$$

P_- the orthogonal projection for L^2 onto $L^2 \ominus H^2$.

All the numerical ranges have the Poncelet property

Theorem (Gau, Wu)

For $T \in \mathcal{S}_n$ and any point $\lambda \in \mathbb{T}$ there is an $(n+1)$-gon inscribed in $\mathbb{T}$ that circumscribes the boundary of $W(T)$ and has λ as a vertex.

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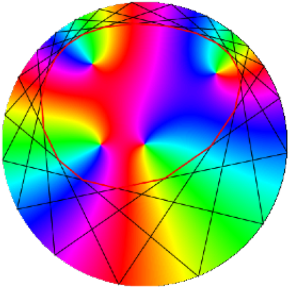

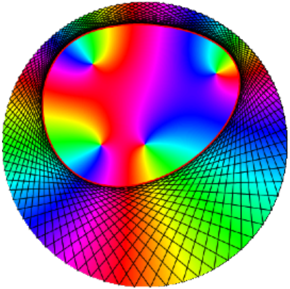

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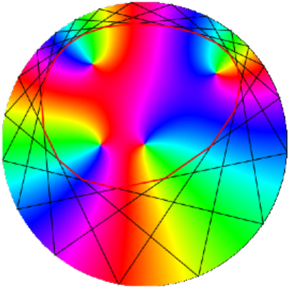

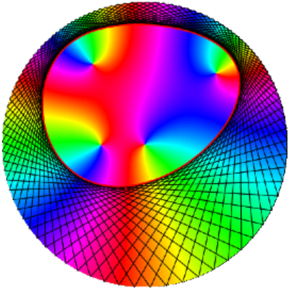

These are not Poncelet ellipses, but they have the Poncelet property. They are *Poncelet curves*.

Important example

$S_B(f) = P_B(S(f))$, $S_B : K_B \to K_B$

When the Blaschke product is $B(z) = z^n$, the matrix representing S_B is the $n \times n$ Jordan block.

Theorem

The numerical range of the $n \times n$ Jordan block is a circular disk of radius $\cos(\pi/(n+1))$.

The boundary of these numerical ranges are all Poncelet circles.

Application of function theory to $T \in \mathcal{S}_n$

Theorem (Special theorem, Gau and Wu, 1995)

$$\overline{W(S_B)} = \bigcap \{ \overline{W(U)} : U \text{ a unitary 1-dilation of } S_B \}.$$

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Theorem (General theorem, Choi and Li, 2001)

$$\overline{W(T)} = \bigcap \{\overline{W(U)} : U \text{ a unitary dilation of } T \text{ on } H \oplus H\}.$$

Gau and Wu's theorem is the "most economical" intersection.

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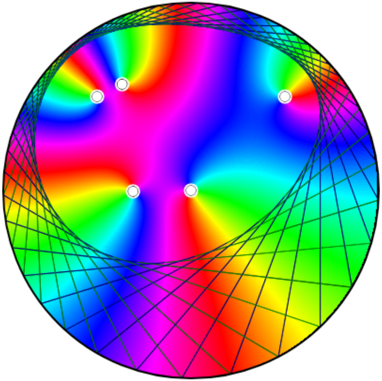


The geometry of infinite Blaschke products

B infinite Blaschke product; $\sum_{n=1}^{\infty}(1-|z_n|)<\infty$

For T a completely nonunitary contraction with a unitary 1-dilation

1. Every eigenvalue of T is in the interior of $W(T)$;
2. $\overline{W(T)}$ has no corners in $\mathbb{D}$.

Orthogonal decompositions of K_I with I inner

To think of S_I as a matrix, we look at it with respect to two decompositions:

Decomposition 1:

$$\mathcal{M}_1 = \mathbb{C}(S^\star I) = \{x(I(z) - I(0))/z\} \text{ and } \mathcal{N}_1 = K_I \ominus \mathcal{M}_1.$$

Decomposition 2:

$$\mathcal{M}_2 = \mathbb{C}(I\,\overline{I(0)} - 1) \text{ and } \mathcal{N}_2 = K_I \ominus \mathcal{M}_2.$$

Computations show:

$$S_I(xS^\star I + w) = x((I\,\overline{I(0)} - 1)I(0) + Sw$$

for $x \in \mathcal{C}$ and $w \in \mathcal{N}_1$.

Infinite Blaschke products and two decompositions

Let S denote the shift operator.

Unitary 1-dilations on $K = H \oplus \mathbb{C}$.

$$S_I = \begin{bmatrix} \lambda & 0 \\ 0 & S \end{bmatrix} \text{ and } U_\lambda = \begin{bmatrix} \lambda & 0 & \alpha\sqrt{1-|\lambda|^2} \\ 0 & S & 0 \\ \beta\sqrt{1-|\lambda|^2} & 0 & -\alpha\beta\overline{\lambda} \end{bmatrix}.$$

If $I(0) = 0$, then $\lambda = 0$.

Theorem (Clark, 1972)

If $I(0) = 0$ all unitary 1-dilations of S_I are equivalent to rank 1 perturbations of S_{zI}.

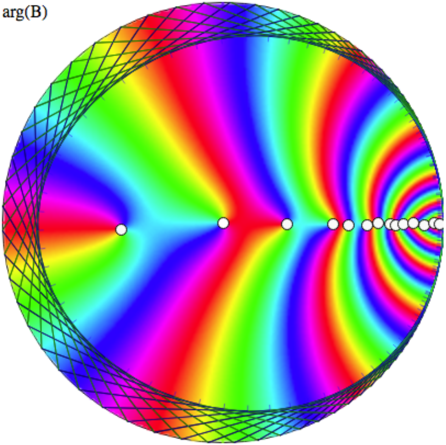
Theorem (Chalendar, G., Partington)

Let B be an infinite Blaschke product. Then the closure of the numerical range of S_B satisfies

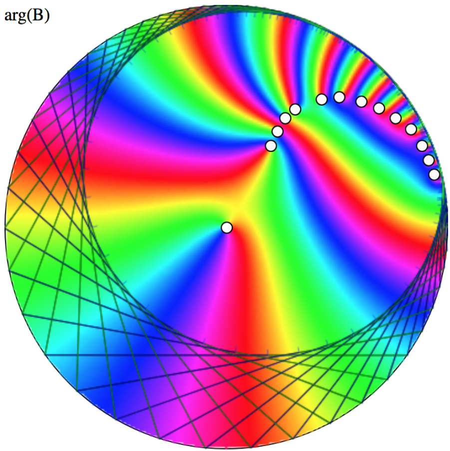
$$\overline{W(S_B)} = \bigcap_{\alpha \in \mathbb{T}} \overline{W(U_\alpha^B)},$$

where the U_α^B are the unitary 1-dilations of S_B (or, equivalently, the rank-1 Clark perturbations of $S_{\hat{B}}$).

For some functions, we get an infinite version of Poncelet's theorem.



An “infinite” Blaschke product with real zeros



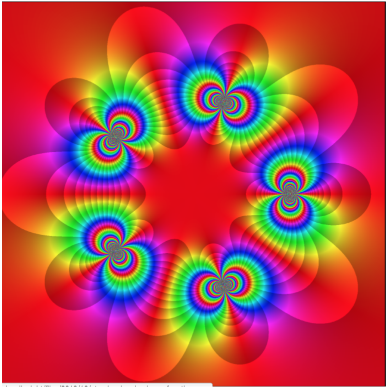
A more general “infinite” Blaschke product

Theorem (Frostman's Theorem)

Let I be an inner function. Let $a \in \mathbb{D}$ and $\varphi_a(z) = \frac{z-a}{1-\overline{a}z}$. Then $\varphi_a \circ I$ is a Blaschke product for almost all $a \in \mathbb{D}$.

Every inner function is a uniform limit of Blaschke products.

An application of Frostman's theorem tells us that $W(S_I)$ has the same property for all I inner.



Modifying $S(z) = exp\left(\frac{z+1}{z-1}\right)$

Further generalizations

Let $D_T = (1 - T^\star T)^{1/2}$ (the defect operator) and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$ (the defect space).

What if the dimension of $\mathcal{D}_T = \mathcal{D}_{T^\star} = n > 1$?

Bercovici and Timotin showed that

$$\overline{W(T)} = \bigcap \{\overline{W(U)} : U \text{ a unitary } n-\text{dilation of } T\}.$$

So that wraps that up...

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Not quite:

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Not quite: (joint work with Kelly Bickel)

$$\mathbb{D}^2 = \{(z_1, z_2) : |z_1|, |z_2| < 1\}$$

$$\mathbb{T}^2 = \{(\tau_1, \tau_2) : |\tau_1|, |\tau_2| = 1\}$$

$$H^2(\mathbb{D}^2) = \{f \in \text{ Hol}(\mathbb{D}^2) : \|f\|_{H^2}^2 = \lim_{r \to 1} \int_{\mathbb{T}^2} |f(r\tau)|^2 d\sigma < \infty\}$$

Θ is inner if $\Theta \in Hol(\mathbb{D}^2)$ and $\lim_{r\to 1} |\Theta(r\tau)| = 1$ for a.e. $\tau \in \mathbb{T}^2$.

$K_\Theta = H^2(\mathbb{D}^2) \ominus \Theta H^2(\mathbb{D}^2)$ is a two variable model space.

$S_{z_1} = P_\Theta M_{z_1}$ and $S_{z_2} = P_\Theta M_{z_2}$ are the compressed shifts.

Rational inner functions

Θ rational inner with $\deg\Theta = (m, n)$ implies there is an (almost) unique polynomial with no zeros on $\mathbb{D}^2$ such that

$$\Theta = \frac{\tilde{p}}{p}, \text{ where } \tilde{p}(z) = z_1^m z_2^n \overline{p(\frac{1}{z_1}, \frac{1}{z_2})}$$

and p and $\tilde{p}$ have no common factors.

Example. A $(1, 1)$ rational inner function is

$$\Theta(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{\overline{a}z_1z_2 + \overline{b}z_2 + \overline{c}z_1 + \overline{d}}{a + bz_1 + cz_2 + dz_1z_2}.$$

There are subspaces E and F of K_Θ such that

$$K_\Theta = \left(\oplus_{j=0}^{\infty} z_1^j E\right) \oplus \left(\oplus_{k=0}^{\infty} z_2^k F\right) = \mathcal{S}_1 \oplus \mathcal{S}_2$$

for subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ invariant under multiplication by z_1 and z_2.

Let $\Theta = \frac{\tilde{p}}{p}$ with deg $\Theta = (m, n)$, $K_\Theta = \mathcal{S}_1 \oplus \mathcal{S}_2$.

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Lemma

Then $S_{z_1}|\mathcal{S}_1 = M_{z_1}$ *and if* $\mathcal{S}_1 \neq \{0\}$, *then* $\overline{W(S_{z_1}|\mathcal{S}_1)} = \overline{\mathbb{D}}$.

So we look at $\tilde{S}_{z_1}|\mathcal{S}_2 = P_{S_2} S_{z_1}|\mathcal{S}_2$.

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$\overrightarrow{f} = (f_1, \dots, f_m)$ with $f_j \in H^2(\mathbb{D})$, Θ rational, inner, degree (m, n), $H^2_2(\mathbb{D})^m = \oplus_{j=1}^m H^2_2(\mathbb{D})$.

Theorem (Bickel, G.)

There exists an $m \times m$ matrix-valued function M_Θ with continuous entries, rational in $\overline{z_2}$ and $\mathcal{U} : H^2_2(\mathbb{D})^m \to \mathcal{S}_2$ unitary such that

$$\tilde{S}_{z_1}|\mathcal{S}_2 = \mathcal{U} T_{M_\Theta} \mathcal{U}^\star,$$

$T_{M_\Theta} : H^2_2(\mathbb{D})^m \to H^2_2(\mathbb{D})^m$ is the matrix valued Toeplitz operator with symbol M_Θ, i.e., $T_{M(\Theta)}(f_1, \dots, f_m) = P_{H^2_2(\mathbb{D})^m}(M(\Theta)\overrightarrow{f})$.

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Theorem

$$\overline{W(\tilde{S}_{z_1}|\mathcal{S}_2)} = Conv\left(\cup_{\tau \in \mathbb{T}} W(M_\Theta(\tau))\right).$$

The right-hand side are things we understand.

Let

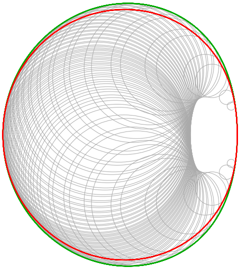
$$\Theta(z) = \left(\frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}\right)\left(\frac{3z_1z_2 - 2z_1 - z_2}{3 - z_1 - 2z_2}\right)$$

be a degree $(2,2)$ inner function. Then

$$M_\Theta(z_2) = \begin{bmatrix} \frac{1}{2-\overline{z_2}} & 0 \\ \frac{-\sqrt{12}(1-\overline{z_2})^2}{2-\overline{z_2})(3-2\overline{z_2})} & \frac{1}{3-2\overline{z_2}} \end{bmatrix}.$$

So $\tilde{S}_{z_1}|\mathcal{S}_2$ is unitarily equivalent to the (matrix-valued) Toeplitz operator with this symbol.

Example: For $\Theta = \theta_1^2$ where θ_1 has a zero on $\mathbb{T}^2$ and $\theta_1 = \frac{\tilde{p}}{p}$ for $p(z) = a - z_1 + cz_2$ with $a, c \neq 0$, Θ is degree $(2,2)$ and so $M_\Theta(\tau)$ is 2×2. The numerical range looks like the convex hull of this:



We can get a formula using envelopes!

Some final comments

• Michel Crouzeix 2006: "Open problems on the numerical range and functional calculus'."

Conjecture (2004): For any polynomial $p \in \mathbb{C}[z]$ and A an $n \times n$ matrix the inequality holds:

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The best constant should be $C = 2$.

Let $p(z) = z$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$LHS = 1 \text{ and } RHS = C \cdot 1/2.$$

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6. (Crouzeix, Palencia) Best constant is between 2 and $1 + \sqrt{2}$.

How sharp is that constant

$A(\Omega)$ continuous functions on $\overline{\Omega}$ holomorphic on Ω.

Lemma

Let T be a bounded operator and Ω be a bounded open set containing the spectrum of T. Suppose that for each $f \in A(\Omega)$ there exists $g \in A(\Omega)$ such that

$$\|g\|_\Omega \leq \|f\|_\Omega \text{ and } \|f(T) + g(T)^\star\| \leq 2\|f\|_\Omega.$$

Then

$$\|f(T)\| \leq (1 + \sqrt{2})\|f\|_\Omega, f \in A(\Omega).$$

Ransford and Schwenninger gave a short proof of this lemma and show that in this lemma, the constant $(1 + \sqrt{2})$ is sharp. Suggest alternate question, for which an affirmative answer would prove the Crouzeix conjecture.

When is the numerical range elliptical

–For S_B with B degree 3, Fujimura showed that the curve formed by looking at points $\hat{B}(z) = zB(z)$ identifies forms an ellipse iff $\hat{B}$ is a composition of two degree 2 Blaschke products.

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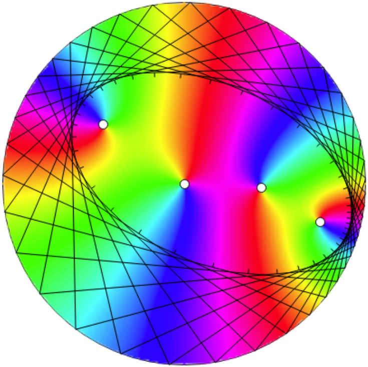

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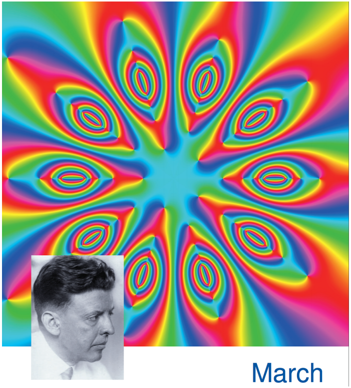
Question. Find necessary and sufficient conditions for $W(S_B)$ to be elliptical.

Thank you!



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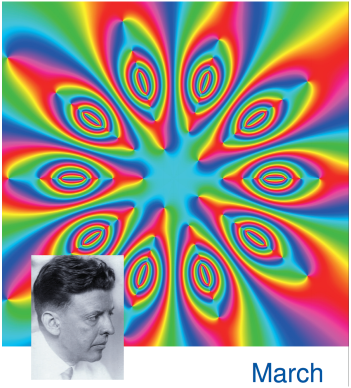

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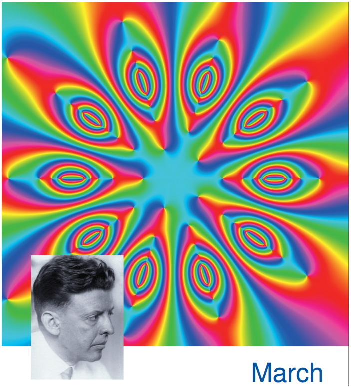

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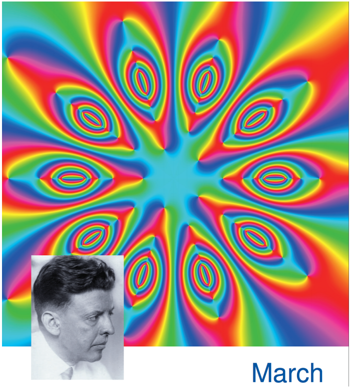

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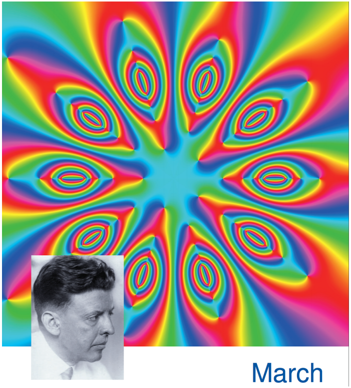

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