# On computability and computational complexity of Julia sets

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# Julia set of a polynomial $f$

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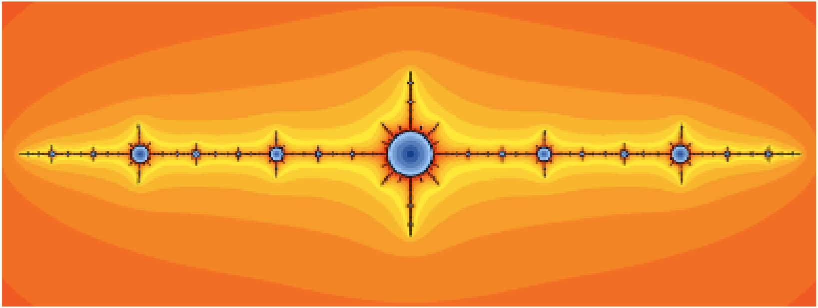

Figure: The airplane map $p(z) = z^2 + c$, $c \approx -1.755$.

# Computability

### Definition

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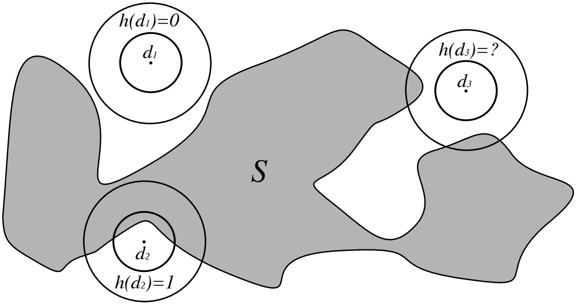
A $2^{-n}$ approximation of a set $S$ can be described using a function

$$h_S(n, z) = \begin{cases} 1, & \text{if } \; d(z, S) \leqslant 2^{-n-1}, \\ 0, & \text{if } \; d(z, S) \geqslant 2 \cdot 2^{-n-1}, \\ 0 \text{ or } 1 & \text{otherwise}, \end{cases}$$

where $n \in \mathbb{N}$ and $z = (i/2^{n+2}, j/2^{n+2}), \; i, j \in \mathbb{Z}$.

# Computational complexity



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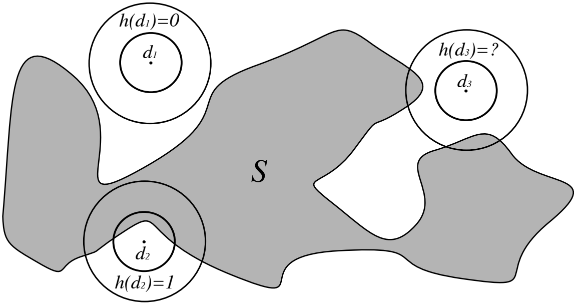


## Definition

*$S \subset \mathbb{R}^2$ is computable in time $t(n)$ if there is an algorithm which computes $h(n, \bullet)$ in time $t(n)$.*

# An oracle

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*A function* $\phi : \mathbb{N} \to \mathbb{D}^n$ *is called an oracle for an element* $x \in \mathbb{R}^n$*, if* $\|\phi(m) - x\| < 2^{-m}$ *for all* $m \in \mathbb{N}$*, where* $\| \cdot \|$ *stands for the Euclidian norm in* $\mathbb{R}^n$*.*

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### Definition

*The Julia set* $J_f$ *of a map* $f$ *is called computable in time* $t(n)$*, if there is an algorithm with an oracle for the values of* $f$*, which computes* $h(n, \bullet)$ *for* $S = J_f$ *in time* $t(n)$*. It is called poly-time if* $t(n)$ *can be bounded by a polynomial.*

# Poly-time computability of hyperbolic Julia sets

A rational map $f$ is called hyperbolic if there is a Riemannian metric $\mu$ on a neighborhood of the Julia set $J_f$ in which $f$ is strictly expanding:

$$\|Df_z(v)\|_\mu > \|v\|_\mu$$

for any $z \in J_f$ and any tangent vector $v$.

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## Proposition (Milnor)

*A rational map $f$ is hyperbolic if and only if every critical orbit of $f$ either converges to an attracting (or a super-attracting) cycle, or is periodic.*

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### Theorem (Braverman 04, Rettinger 05)

*For any $d \geqslant 2$ there exists a Turing Machine with an oracle for the coefficients of a rational map of degree $d$ which computes the Julia set of every hyperbolic rational map in polynomial time.*

# Distance estimator

Let $f(z)$ be a hyperbolic rational map. Compute a closed neighborhood $U$ of $J_f$ which does not contain any attracting periodic points or critical points and such that $\mu$ is expanding with constant $\gamma > 1$ on $U$. Fix sufficiently large number $C$ (of order $\log 2 / \log \gamma$).

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- given a dyadic point $z$ and $n \in \mathbb{N}$ compute approximate values of $z_k = f^k(z)$, $1 \leqslant k \leqslant Cn$;

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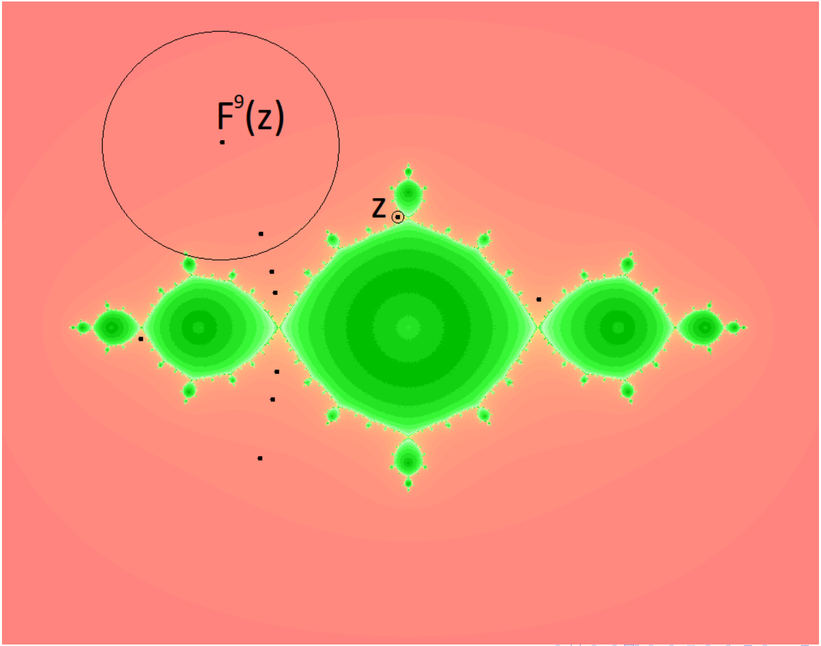
- if $z_k \in U$ for all $1 \leqslant k \leqslant Cn$ then $\mathrm{d}(z, J_f) < 2^{-n}$;
- if $z_k \notin U$ for some $1 \leqslant k \leqslant Cn$ then by Koebe distortion Theorem up to a constant factor

$$\mathrm{d}(z, J_f) \approx \frac{\mathrm{d}(z_k, J_f)}{|DF^k(z)|} \approx \frac{1}{|DF^k(z)|}.$$

# Distance estimator



# Poly-time computability of parabolic Julia sets

For a holomorphic map $f$ a periodic point $z_0$ of period $p$ is parabolic if $Df^p(z_0) = \exp(2\pi i\theta), \theta \in \mathbb{Q}$, and $f^p$ is not conjugated to a rotation near $z_0$.

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### Theorem (Braverman 06)

*For any $d \geqslant 2$ there exists a Turing Machine $\mathcal{M}$ with an oracle for the coefficients of a rational map $f$ of degree $d$ such that the following is true. Given that every critical orbit of $f$ converges either to an attracting or to a parabolic orbit, $\mathcal{M}$ computes $J_f$ in polynomial time.*

# Dynamics near parabolic points

For simplicity, assume $f(z_0) = z_0$ and $Df(z_0) = 1$.

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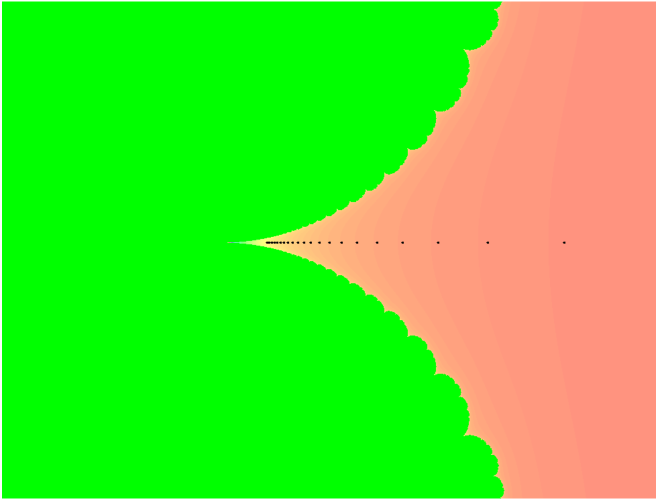

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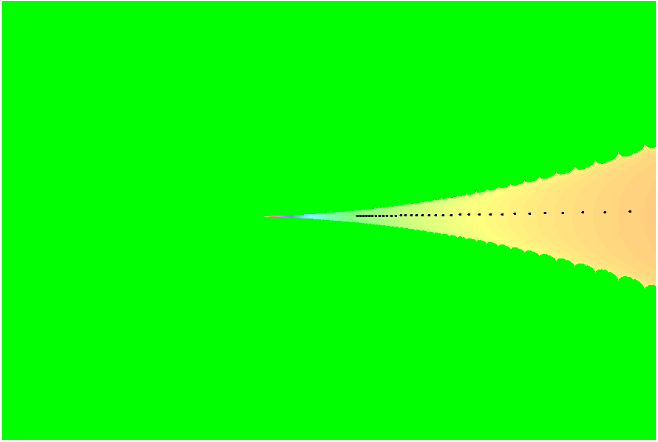

# Speeding up the dynamics

For simplicity, assume $f(z_0) = z_0$ and $Df(z_0) = 1$.

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**Solution 2:** Fatou coordinates $\phi^i_{a,r}$ conjugate $f$ to $z \to z + 1$ near $z_0$; $\phi^i_{a,r}$ can by approximated effectively by the formal solutions of the Fatou coordinate equation $\phi \circ f(z) = z + 1$ (Dudko-Sauzin 14).

# Siegel periodic points

For a holomorphic map $f$ a periodic point $z_0$ of period $p$ is called *Siegel* if $Df^p(z_0) = \exp(2\pi i\theta), \theta \in \mathbb{R} \setminus \mathbb{Q}$, and $f^p$ is conjugated (by a conformal map) to a rotation near $z_0$. The maximal domain around $z_0$ on which such conjugacy exists is called *Siegel disk*.

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Consider $P_\theta(z) = \exp(2\pi i\theta)z + z^2, \;\; \theta \in [0,1)$. Let $p_n/q_n$ be the sequence of the closest rational approximations of $\theta$ and

$$B(\theta) = \sum \frac{\log(q_{n+1})}{q_n}.$$

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### Theorem (Brjuno 72, Yoccoz 81)

*Origin is a Siegel point for $P_\theta$ iff $B(\theta) < \infty$.*

# Computability and complexity of Siegel Julia sets

### Theorem (Braverman-Yampolsky 06, 09)

*There exists $P_\theta$ with a Siegel fixed point at the origin such that $J_{P_\theta}$ is not computable. Moreover, $\theta$ can be chosen computable and such that $J_{P_\theta}$ is locally connected.*

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Let $\Delta(\theta)$ be the Siegel disk of $P_\theta$, $\rho(\theta) = \inf_{z \in \partial\Delta(\theta)} |z|$ be the inner radius of $\Delta(\theta)$ and $r(\theta)$ be the conformal radius of $\Delta(\theta)$.

# Constructing non-computable Siegel Julia sets

Theorem (Binder-Braverman-Yampolsky 06)

*The following statements are equivalent:*

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Take $r \in (0, 0.1]$ right-computable but not computable. Let $\theta$ be such that $r(\theta) = r$. Then $J_{P_\theta}$ is not computalbe.

# Poly-time computability of the Feigenbaum Julia set

Let $F$ be the fixed point of the period-doubling renormalization (also referred to as the Feigenbaum map). The map $F$ is the solution of the Cvitanović-Feigenbaum equation:

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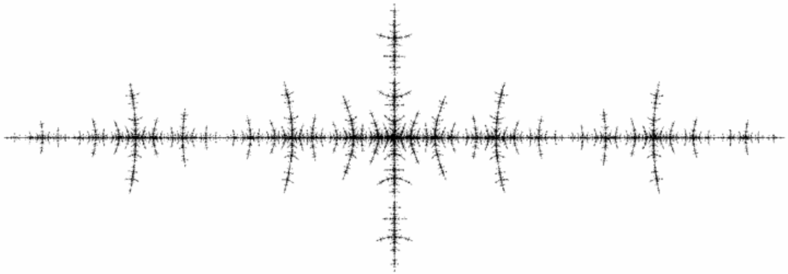
**Theorem (Dudko-Yampolsky 16)**

*The Julia set $J_F$ is poly-time computable.*

# The Feigenbaum Julia set



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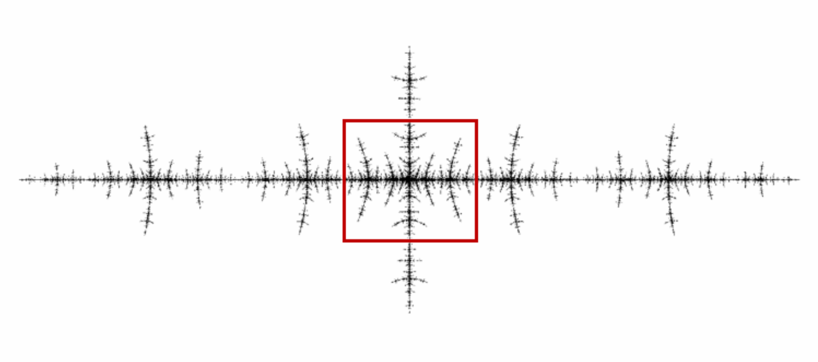

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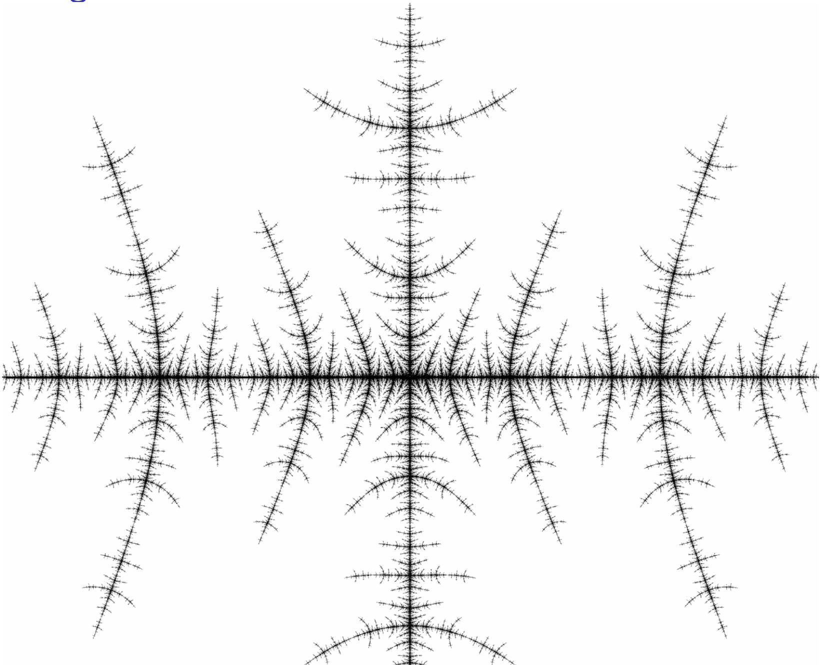

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**Solution:** the dynamics can be speeded up by:

$$F^{2^k}(z) = (-\lambda)^k F(z/\lambda^k), \quad |z| < C\lambda^k.$$

For $z$ with $\mathrm{d}(z, J_F) \approx 2^{-n}$ polynomial number of speeded up iterations is sufficient to escape $\epsilon$-neighborhood of $J_F$. Moreover, the distortion of the iterate is bounded near $z$.

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We used the algorithms designed for computing $J_F$ in the computer-assisted proof of

### Theorem (Dudko-Sutherland 17)

*The Julia set $J_F$ has Hausdorff dimension less than two (and therefore its Lebesgue area is zero).*

# Collet-Eckmann maps

**Definition**

*A non-hyperbolic rational map $f$ is called Collet-Eckmann if there exist constants $C, \gamma > 0$ such that the following holds: for any critical point $c \in J_f$ of $f$ whose forward orbit does not contain any critical points one has:*

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### Theorem (Avila-Moreira 05)

*For almost every real parameter $c$ the map $f_c(z) = z^2 + c$ is either Collet-Eckmann or hyperbolic.*

# Exponential Shrinking of Components

**Definition**

*A rational map $f$ satisfies Exponential Shrinking of Components (ESC) condition if there exists $\lambda < 1$ and $r > 0$ such that for every $n \in \mathbb{N}$, any $x \in J_f$ and any connected component $W$ of $f^{-n}(U_r(x))$ one has $\mathrm{diam}(W) < \lambda^n$.*

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**Theorem (Przytycki–Rivera-Letelier–Smirnov 03)**

*Collet-Eckmann condition implies Exponential Shrinking of Components condition.*

# Poly-time computability of CE Julia sets

Theorem (Dudko-Yampolsky 17)

*For each $d \geqslant 2$ there exists an oracle Turing Machine $\mathcal{M}$ with an oracle for the coefficients of a rational map $f$ satisfying ESC, which, given a certain non-uniform information, computes $J_f$ in polynomial time.*

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### Corollary

*For almost every real value of the parameter $c$, the Julia set $J_c$ is poly-time.*

# Distance estimator for CE maps

By definition, for an ESC map $f$ one can find $\epsilon > 0$ and $C > 0$ such that for any point $z$ with $\mathrm{d}(z, J_f) \approx 2^{-n}$ one has

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**Solution:** we show that $f^i(z), 0 \leqslant i \leqslant Cn$, approach critical points at most $K\sqrt{n}$ times and the distortion of $f^{Cn}$ near $z$ is bounded by $M^{\sqrt{n}}$. This allows to estimate $\mathrm{d}(z, J_F)$ up to $M^{\sqrt{n}}$.

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- There exists a computable $c \in \mathbb{C}$ and a computable angle $\alpha \in \mathbb{R}$ such that the impression of the external angle corresponding to $\alpha$ is non-computable (Binder-Rojas-Yampolsky 15).
- There exists a (natural) family of cubic polynomials for which the connectedness locus (Mandelbrot-like set) is non-computable (Coronel-Rojas-Yampolsky 17).

# Open questions

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- Are Julia sets of all Feigenbaum maps (infinitely renormalizable with bounded combinatorics and a priori bounds) poly-time?
- What can be said about computability and computational complexity of Julia sets (or escaping, or fast escaping sets) of transcendental entire maps?

Thank you!