Weighted Harmonic Mappings in the Unit Disk

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Outline of Report

1. Energy functionals and harmonic mappings
2. Dirichlet boundary value problem
3. Green function and Poisson kernel
4. Three kinds of representation theorems
5. Three applications

Definition of σ-harmonic mappings [Garabedian]

Let σ be a smooth positive metric density on Ω. If a mapping u of Ω onto Ω' is a critical point of the energy functional

$$E_1^\sigma(z) = \iint_\Omega \sigma(z)|u_z|^2 dxdy,$$

or

$$E_2^\sigma(z) = \iint_\Omega \sigma(z)|u_{\bar{z}}|^2 dxdy,$$

then it satisfies the Euler-Lagrange equation

$$(\sigma(z)u_z)_{\bar{z}} = 0, \tag{1}$$

or

$$(\sigma(z)u_{\bar{z}})_z = 0, \tag{2}$$

respectively. We note that we can get them from first variation.

α-harmonic mapping, $\bar{\alpha}$-harmonic mapping

Let $\sigma(z) = (1 - |z|^2)^{-\alpha}$ in the unit disk, herein $\alpha > -1$. We write

$$L_\alpha u = ((1 - |z|^2)^{-\alpha} u_z)_{\bar{z}}$$

and

$$\overline{L_\alpha} u = ((1 - |z|^2)^{-\alpha} u_{\bar{z}})_z.$$

If *f* satisfies that

$$-L_\alpha u = 0,$$

or

$$-\overline{L_\alpha} u = 0,$$

then we call it a α*-harmonic mapping* or a $\bar{\alpha}$*-harmonic mapping*, respectively.

T_α-harmonic mapping [Olofsson]

If u satisfies that

$$T_\alpha u = \frac{1}{2}L_\alpha u + \frac{1}{2}\overline{L_\alpha} u - \frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1}u$$
$$= \frac{1}{2}((1-|z|^2)^{-\alpha}u_z)_{\bar z} + \frac{1}{2}((1-|z|^2)^{-\alpha}u_z)_{\bar z} - \frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1}u = 0$$

then we call it a T_α*-harmonic mapping*.

If $\alpha = 2$, then $T_2 u = 0$ can be simplified as

$$\bar z \partial_{\bar z} u(z) + z\partial_z u(z) + (1-|z|^2)\partial_z\partial_{\bar z} u(z) = u(z). \tag{3}$$

Dirichlet boundary value problem

The *associated Dirichlet boundary value problem* of the α-harmonic equation or $\bar{\alpha}$-harmonic equation is the following problem:

$$\begin{cases} -L_\alpha v = g & \text{in } \mathbb{D}, \\ v = f & \text{on } \mathbb{T}, \end{cases}$$

or

$$\begin{cases} -\overline{L_\alpha} v = g & \text{in } \mathbb{D}, \\ v = f & \text{on } \mathbb{T}, \end{cases}$$

here $g \in C(\mathbb{D})$, $f \in L^1(\mathbb{T})$, and the boundary condition is to be understood as $u(re^{i\theta}) \to f$ in $L^1(\mathbb{T})$ when $r \to 1$.

Green function

Green's function $G_\alpha(z, w)$ of the weighted Laplacian operator L_α is the function defined on $\Omega \times \Omega$ solving for fixed $w \in \Omega$:

$$(1) \quad -L_\alpha G_\alpha(z, w) = \delta(z - w),$$
$$(2) \quad G_\alpha(z, w) \to 0, \text{ when } z \to \zeta \in \partial\Omega,$$

where $\delta(z - w)$ is the Dirac function in z with support at w and similarly if we interchange the roles of z and w.

A route to get Green function for a given PDE

Find a radial solution (fundamental solution) of the PDE, that is, turn the PDE into a differential equation in r and then solve it,

Deform the fundamental solution by Möbius transformation $g(z, w) = c\frac{z-w}{1-\bar{w}z}$ with $|c| = 1$,

Normalize the deformation solution by Green theorem to get Green function.

Green function

For example, Green function for L_α
Let $F = F(|z|)$, then

$$L_\sigma(F) = \overline{L}_\sigma(F) = \frac{\partial\sigma(r)}{\partial r}\frac{\partial}{\partial r}(F) + \sigma(r)\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}(F)). \tag{4}$$

$$F(r) = \int \frac{c}{r\rho(r)}dr, \qquad F(r) = c\int \frac{(1-r^2)^\alpha}{r}dr,$$

$$\Phi(z) = \frac{1}{2\pi}\int_{|z|}^{1} \frac{(1-r^2)^\alpha}{r}dr$$

$$G_\alpha(z,w) = (1-\bar{z}w)^\alpha \Phi(g(z,w))$$

Poisson kernel

Write

$$P_\alpha(z) = P_r^\alpha(\theta) = \frac{1}{2\pi}\frac{(1-|z|^2)^{\alpha+1}}{(1-\bar{z})^{\alpha+1}(1-z)}, \quad z = re^{i\theta}. \tag{5}$$

Then it is a $\bar{\alpha}$-harmonic mapping and we call it the $\bar{\alpha}$-*Poisson kernel*. Specially, if $\alpha = 0$, then $P_r^\alpha(\theta)$ is the classical Poisson kernel

$$P_r(\theta) = \frac{1}{2\pi}\frac{1-|z|^2}{|1-z|^2}, \quad z = re^{i\theta}.$$

Integral Representation [Oloffson, Behm, Chen and Kalaj]

$$v(w) = u(w) + G_\alpha[g](w),$$

where

$$u(w) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^{\alpha+1}}{(1-z\bar{w})^{\alpha+1}(1-\bar{z}w)} f(z)d\theta,$$

$$G_\alpha[g](w) = \int_{\mathbb{D}} G_\alpha(z,w)g(z)dxdy,$$

$$G_\alpha(z,w) = \frac{(1-\bar{z}w)^\alpha h\circ q}{2\pi}, \quad z \neq w,$$

$$h(r) = \frac{1}{2}\int_0^{1-r^2} \frac{t^\alpha}{1-t}dt, \quad q = q(z,w) = |\frac{z-w}{1-\bar{w}z}|.$$

Series representation [Chen, Kalaj]

An $\bar{\alpha}$-harmonic mapping $u(z)$ in the unit disk $\mathbb{D}$ can be represented by the Fourier series $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int} + \sum_{n=1}^{\infty} \overline{b_n} e^{-int}$. Then

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} [\sum_{k=0}^{\infty} \binom{k+n-1}{k} (1-|z|^2)^k - (1-|z|^2)^{\alpha+1} \sum_{k=0}^{\infty} \binom{n+\alpha+k}{k+\alpha+1} (1-|z|^2)^k] \bar{z}^n, \quad (6)$$

$$u(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} [(1-|z|^2)^{\alpha+1} \sum_{k=0}^{\infty} \frac{(\alpha+1)_{(k+n)} |z|^{2k}}{(k+n)!}] \bar{z}^n, \quad (7)$$

where $(\alpha+1)_k = (\alpha+1)(\alpha+2)\cdots(\alpha+k)$ and $(\alpha+1)_0 = 1$.

Analytic representation [Chen, Kalaj]

Assume that a Fourier series $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int} + \sum_{n=1}^{\infty} \overline{b_n} e^{-int}$. Let $f(w) = h(w) + \overline{g(w)}$, $h(w) = \sum_{n=0}^{\infty} a_n w^n$ and $g(w) = \sum_{n=1}^{\infty} b_n w^n$. If α is a nonnegative integer m and u has a boundary function $f \in L^1(T)$, then

$$u(w) = h(w) + \sum_{k=0}^{m} (1 - |w|^2)^k \overline{I_k}, \tag{8}$$

where I_k satisfies the recurrence formula

$$I_k = \frac{(k-1)I_{k-1} + wI'_{k-1}}{k}, \quad k = 1, 2, \cdots, m, \tag{9}$$

and $I_0 = g(w)$.

Analytic representation

when $\alpha = 1$,

$$u(w) = h(w) + \overline{g(w)} + (1 - |w|^2)\overline{wg'(w)};$$

when $\alpha = 2$,

$$u(w) = h(w) + \overline{g(w)} + (1 - |w|^2)\overline{wg'(w)} + (1 - |w|^2)^2 \overline{[wg'(w) + \frac{w^2 g''(w)}{2}]}.$$

A T_2-harmonic mapping has a representation

$$u(z) = \frac{1}{2}(1 - |z|^2)(zg_z(z) + \bar{z}g_{\bar{z}}(z)) + \frac{1}{2}(1 + |z|^2)g(z), \quad z \in \mathbb{D}, \quad (10)$$

where $g(z)$ is a harmonic mapping in $\mathbb{D}$.

Radó-Kneser-Choquet Theorem

Radó-Kneser-Choquet Theorem

If $\Omega \subset \mathbb{C}$ is a bounded convex domain whose boundary is a Jordan curve Γ and f is a homeomorphism of the unit circle $\mathbb{T}$ onto Γ, then its harmonic extension

$$u(w) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1-|w|^2}{|e^{it}-w|^2} f(e^{it})dt$$

is univalent in $\mathbb{D}$ and defines a Euclidean harmonic mapping of $\mathbb{D}$ onto Ω.

D. Kalaj, G.J. Martin

2017, Radó – Kneser – Choquet theorem for harmonic mappings between surfaces, CVPDE.
2016, Harmonic degree 1 maps are diffeomorphisms: Lewy's theorem for curved metrics, TAMS.

Counterexample for Radó-Kneser-Choquet Theorem

Counterexample Theorem [Chen, Kalaj]

Assume that $f(e^{it}) = e^{it} + \frac{1}{s}e^{-2it}$, $s \geq 2$. Let u be an $\bar{\alpha}$-harmonic mapping with the boundary function f. Then the following assertions are true.

(1) f maps the unit circle $\mathbb{T}$ onto a convex Jordan curve if and only if $s \geq 4$.

(2) When α is equal to 1, the Jacobian $J_u > 0$ if $s \geq 4$ hence u is globally univalent on $\mathbb{D}$.

(3) When α is equal to 2, u is not univalent on $\mathbb{D}$ if $s = 4$.

Lipschitz Continuity

Eulcidean harmonic quasiconformal mappings [Pavlovic, Kalaj]

$\bar{\alpha}$-harmonic quasiconformal mappings [Chen]

Assume that $v \in V_{\mathbb{D}\to\Omega}[g]$ with the representation $v(w) = u(w) + G_\alpha[g](w)$. Then the following conditions are equivalent.
(a) v is a (K, K')-quasiconformal mapping and $|\frac{\partial u}{\partial r}| \leq L$ on $\mathbb{D}$, here L is a constant.
(b) v is Lipschitz continuous with the Euclidean metric.
(c) u is Lipschitz continuous with the Euclidean metric.
(d) f is absolutely continuous on $\mathbb{T}$, $f' \in L^\infty(\mathbb{T})$ and the following integral

$$\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|w|^2)^\alpha}{(1-z\bar{w})^\alpha} \frac{(w\bar{z}-\bar{w}z)/i}{|z-w|^2}[f(e^{i\theta})]'_\theta d\theta$$

is bounded, here $z = e^{i\theta}$.

Decomposition of Poisson kernel for T_α

The following kernel function

$$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}$$

is a T_α-function. Here, $c_\alpha = \frac{[\Gamma(\frac{\alpha}{2}+1)]^2}{\Gamma(1+\alpha)}$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $s > 0$ is the standard Gamma function.

$$\begin{aligned} K_2(z) =& \frac{1}{2}\frac{(1-|z|^2)^3}{|1-z|^4} \\ =& -\frac{1}{2}(1-|z|^2) + \frac{1}{2}\frac{1}{(1-z)^2} + \frac{1}{2}\frac{1}{(1-\bar{z})^2} \\ & -\frac{1}{2}\frac{z^2|z|^2}{(1-z)^2} - \frac{1}{2}\frac{\bar{z}^2|z|^2}{(1-\bar{z})^2}, \end{aligned} \tag{11}$$

Schwarz Lemma

Schwarz, Heinz, Kalaj, Vuorinen, H.H. Chen

Boundary functions

$$f_a(e^{i\theta}) = \begin{cases} 1, & |\theta - \varphi| < \frac{(1+2a)\pi}{2} \\ -1, & \frac{(1+2a)\pi}{2} < |\theta - \varphi| < \pi \end{cases} \quad (12)$$

and

$$f_{-a}(e^{i\theta}) = \begin{cases} 1, & |\theta - \varphi| < \frac{(1-2a)\pi}{2} \\ -1, & \frac{(1-2a)\pi}{2} < |\theta - \varphi| < \pi \end{cases} \quad (13)$$

Schwarz Lemma [Chen, Li]

Let $F:\mathbb{D}\to I$ be a T_2-harmonic function. Then for any $z=re^{i\varphi}\in\mathbb{D}$, we have

$$\tilde{F}_a(-r)\le F(z)+(1-r^2)F(0)\le \tilde{F}_a(r) \tag{14}$$

Here

$$\tilde{F}_a(r)=\frac{2}{\pi}\Big[\frac{r(1-r^2)\cos a\pi}{1+r^2+2r\sin a\pi}+\arctan\Big(\frac{r+\sin a\pi}{\cos a\pi}\Big) +r^2\arctan\Big(\frac{r\cos a\pi}{1+r\sin a\pi}\Big)\Big].$$

Equality on the right (resp., left-) hand side holds for the function

$$F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_2(ze^{-i\theta})f_a(e^{i\theta})d\theta \tag{15}$$

$$(resp., F(z)=-\frac{1}{2\pi}\int_{-\pi}^{\pi}K_2(ze^{-i\theta})f_{-a}(e^{i\theta})d\theta) \tag{16}$$

and all $z\in\mathbb{D}$

Schwarz Lemma

$F(0) = 0$

Let $F : \mathbb{D} \to I$ be a T_2-harmonic functions. Then for any $z = re^{i\varphi} \in \mathbb{D}$ with $F(0) = 0$, we have

$$|F(z)| \leq \frac{2}{\pi}[(1 + r^2)\arctan r + \frac{r(1 - r^2)}{1 + r^2}]$$

Equality holds for the function

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_2(ze^{-i\theta})f_0(e^{i\theta})d\theta$$

and all $z \in \mathbb{D}$.

Thank you for your attentions!