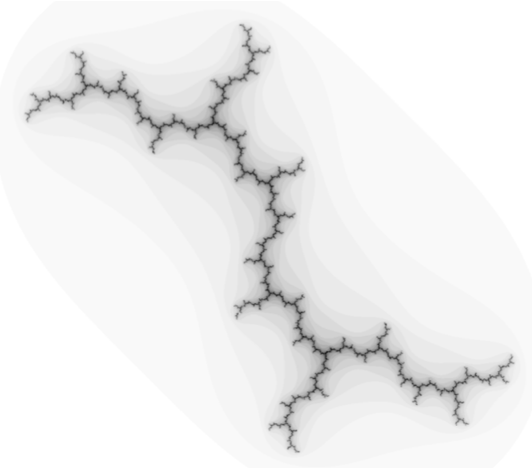
Trees in Dynamics and Probability

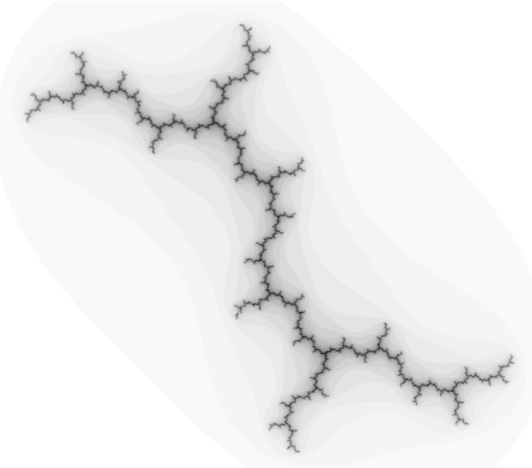
Mario Bonk (UCLA)
joint work with Huy Tran (TU Berlin) and with
Daniel Meyer (U. Liverpool)

New Developments in Complex Analysis and Function Theory
University of Crete, July 2018

What do we see?



What do we see?



The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ (= set of points with bounded orbit under iteration).

What do we *really* see?

The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ is:

- a *dendrite*, i.e., a locally connected continuum with empty interior that does not separate the plane.

  Follows from non-trivial, but standard facts in complex dynamics, because P is *postcritically-finite* and has no finite periodic critical points:

  $$0 \longrightarrow i \longrightarrow -1+i \rightleftarrows -i$$

- a *continuum tree*, i.e., a locally connected, connected, compact metric space s.t. any two points can be joined by a unique arc.

What do we *really* see?

The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ is:

- a *dendrite*, i.e., a locally connected continuum with empty interior that does not separate the plane.

  Follows from non-trivial, but standard facts in complex dynamics, because P is *postcritically-finite* and has no finite periodic critical points:

  $$0 \longrightarrow i \longrightarrow -1+i \rightleftarrows -i$$

- a *continuum tree*, i.e., a locally connected, connected, compact metric space s.t. any two points can be joined by a unique arc.

What do we *really* see?

The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ is:

- a *dendrite*, i.e., a locally connected continuum with empty interior that does not separate the plane.

  Follows from non-trivial, but standard facts in complex dynamics, because P is *postcritically-finite* and has no finite periodic critical points:

  $$0 \longrightarrow i \longrightarrow -1+i \rightleftarrows -i$$

- a *continuum tree*, i.e., a locally connected, connected, compact metric space s.t. any two points can be joined by a unique arc.

What do we *really* see?

The Julia set $\mathcal{J}(P)$ of $P(z) = z^2 + i$ is:

- a *dendrite*, i.e., a locally connected continuum with empty interior that does not separate the plane.

  Follows from non-trivial, but standard facts in complex dynamics, because P is *postcritically-finite* and has no finite periodic critical points:

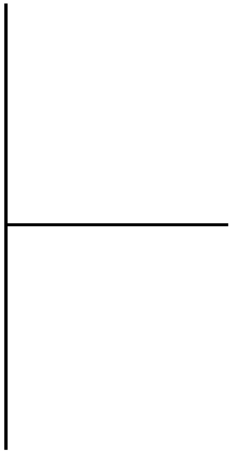
  $$0 \longrightarrow i \longrightarrow -1+i \rightleftarrows -i$$

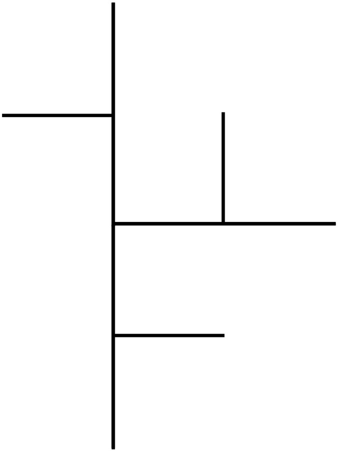
- a *continuum tree*, i.e., a locally connected, connected, compact metric space s.t. any two points can be joined by a unique arc.

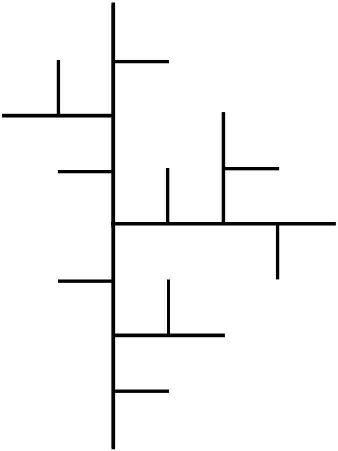
(Continuum) trees appear in various contexts:

- as Julia sets,
- as attractors of iterated function systems (e.g., the CSST=continuum self-similar tree),
- in probabilistic models (e.g., the CRT=continuum random tree).

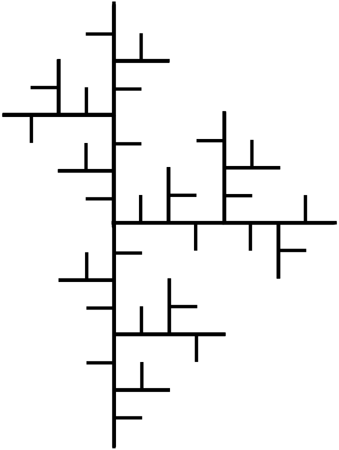
CSST $\mathbb{T}$ (=continuum self-similar tree)



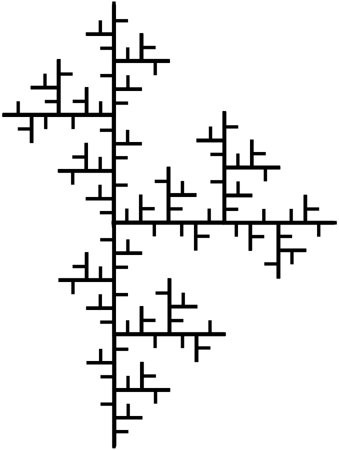




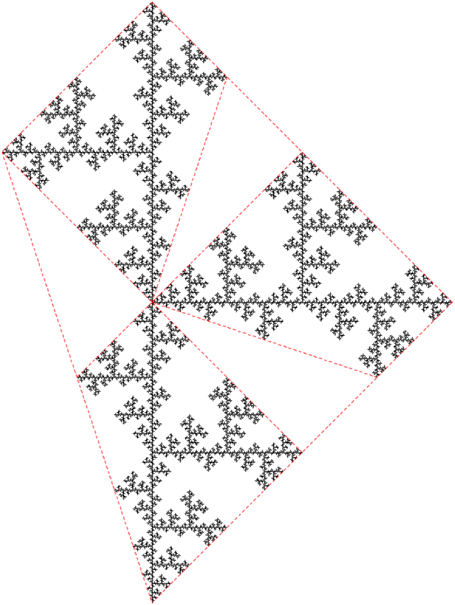
CSST $\mathbb{T}$



CSST $\mathbb{T}$



CSST $\mathbb{T}$



The CSST $\mathbb{T}$ is:

- a geodesic continuum tree (as an abstract metric space).
- an attractor of an iterated function system (as a subset of the plane). Define

$$f_1(z) = \frac{1}{2}z - \frac{i}{2}, \quad f_2(z) = -\frac{1}{2}\bar{z} + \frac{i}{2}, \quad f_3(z) = \frac{i}{2}\bar{z} + \frac{1}{2},$$

Then $\mathbb{T} \subseteq \mathbb{C}$ is the unique non-empty compact set satisfying

$$\mathbb{T} = f_1(\mathbb{T}) \cup f_2(\mathbb{T}) \cup f_3(\mathbb{T}).$$

So $\mathbb{T}$ is the attractor of the iterated function system $\{f_1, f_2, f_3\}$ in the plane.

Topological characterization of the CSST

Theorem (B.-Huy Tran 2018; folklore)

A continuum tree T is homeomorphic to the CCST $\mathbb{T}$ iff all branch points of T have order 3 and they are dense in T.

Proof: $\Rightarrow$: Looks obvious, but is somewhat involved if one defines $\mathbb{T}$ as an attractor of an iterated function system.

$\Leftarrow$: For each level $n \in \mathbb{N}$ carefully cut T into 3^n pieces.
Label pieces by words in a finite alphabet to align with pieces of $\mathbb{T}$.
Use general lemma to obtain a homeomorphism. □

Theorem (B.-Huy Tran 2018; folklore)

A continuum tree T is homeomorphic to the CCST $\mathbb{T}$ iff all branch points of T have order 3 and they are dense in T.

Proof: $\Rightarrow$: Looks obvious, but is somewhat involved if one defines $\mathbb{T}$ as an attractor of an iterated function system.

$\Leftarrow$: For each level $n \in \mathbb{N}$ carefully cut T into 3^n pieces.
Label pieces by words in a finite alphabet to align with pieces of $\mathbb{T}$.
Use general lemma to obtain a homeomorphism. $\square$

Topological characterization of the CSST

Theorem (B.-Huy Tran 2018; folklore)

A continuum tree T is homeomorphic to the CCST $\mathbb{T}$ iff all branch points of T have order 3 and they are dense in T.

Proof: $\Rightarrow$: Looks obvious, but is somewhat involved if one defines $\mathbb{T}$ as an attractor of an iterated function system.

$\Leftarrow$: For each level $n \in \mathbb{N}$ carefully cut T into 3^n pieces.
Label pieces by words in a finite alphabet to align with pieces of $\mathbb{T}$.
Use general lemma to obtain a homeomorphism. □

Lemma providing a homeomorphism

Let (X, d_X) and (Y, d_Y) be compact metric spaces. Suppose that for each level $n \in \mathbb{N}$, the space X admits a decomposition $X = \bigcup_{i=1}^{M_n} X_{n,i}$ as a finite union of non-empty compact subsets $X_{n,i}$ with the following properties:

(i) Each set $X_{n+1,j}$ is the subset of some set $X_{n,i}$.

(ii) Each set $X_{n,i}$ is equal to the union of some of the sets $X_{n+1,j}$.

(iii) $\max_{1 \le i \le M_n} \operatorname{diam}(X_{n,i}) \to 0$ as $n \to \infty$.

Suppose that for $n \in \mathbb{N}$ the space Y admits similar decompositions $Y = \bigcup_{i=1}^{M_n} Y_{n,i}$ with properties analogous to (i)–(iii) such that

$$X_{n+1,j} \subseteq X_{n,i} \text{ if and only if } Y_{n+1,j} \subseteq Y_{n,i} \tag{1}$$

and

$$X_{n,i} \cap X_{n,j} \neq \emptyset \text{ if and only if } Y_{n,i} \cap Y_{n,j} \neq \emptyset \tag{2}$$

for all n, i, j.

Then there exists a unique homeomorphism $f\colon X \to Y$ such that $f(X_{n,i}) = Y_{n,i}$ for all n and i. In particular, the spaces X and Y are homeomorphic.

Decomposition of T into pieces

There are two type of pieces X: *end-pieces* (with one distinguished leaf) and *arc-pieces* (with two distinguished leaves and hence a distinguished arc $\alpha \subseteq X$).

Each end-piece X is cut into three children by using a branch point $p \in X$ of largest *weight* $w(p)$ in T.

Each arc piece X is cut into three children by using a branch point $p \in \alpha$ that is contained in the distinguished arc α of X and has largest weight $w(p)$ in T among these branch points.

$w(p) =$ diameter of third largest component of $T \backslash \{p\}$.

Decomposition of T into pieces

There are two type of pieces X: *end-pieces* (with one distinguished leaf) and *arc-pieces* (with two distinguished leaves and hence a distinguished arc $\alpha \subseteq X$).

Each end-piece X is cut into three children by using a branch point $p \in X$ of largest *weight* $w(p)$ in T.

Each arc piece X is cut into three children by using a branch point $p \in \alpha$ that is contained in the distinguished arc α of X and has largest weight $w(p)$ in T among these branch points.

$w(p) =$ diameter of third largest component of $T \setminus \{p\}$.

Decomposition of T into pieces

There are two type of pieces X: *end-pieces* (with one distinguished leaf) and *arc-pieces* (with two distinguished leaves and hence a distinguished arc $\alpha \subseteq X$).

Each end-piece X is cut into three children by using a branch point $p \in X$ of largest *weight* $w(p)$ in T.

Each arc piece X is cut into three children by using a branch point $p \in \alpha$ that is contained in the distinguished arc α of X and has largest weight $w(p)$ in T among these branch points.

$w(p) =$ diameter of third largest component of $T \backslash \{p\}$.

Decomposition of T into pieces

There are two type of pieces X: *end-pieces* (with one distinguished leaf) and *arc-pieces* (with two distinguished leaves and hence a distinguished arc $\alpha \subseteq X$).

Each end-piece X is cut into three children by using a branch point $p \in X$ of largest *weight* $w(p)$ in T.

Each arc piece X is cut into three children by using a branch point $p \in \alpha$ that is contained in the distinguished arc α of X and has largest weight $w(p)$ in T among these branch points.

$w(p) =$ diameter of third largest component of $T \backslash \{p\}$.

CRT (=continuum random tree)

The CRT is a random geodesic continuum tree $T = T(\omega)$ constructed from Brownian excursion.

If $e = e(\omega)\colon [0,1] \to [0,\infty)$ is a sample of Brownian excursion, define

$$d(s,t) = e(s) + e(t) - 2 \min_{u \in [s,t]} e(u).$$

Then $T = [0,1]/\sim$, where

$$\begin{aligned} s \sim t :&\Leftrightarrow d(s,t) = 0 \\ &\Leftrightarrow e(s) = e(t) = \min_{u \in [s,t]} e(u). \end{aligned}$$

CRT (=continuum random tree)

The CRT is a random geodesic continuum tree $T = T(\omega)$ constructed from Brownian excursion.

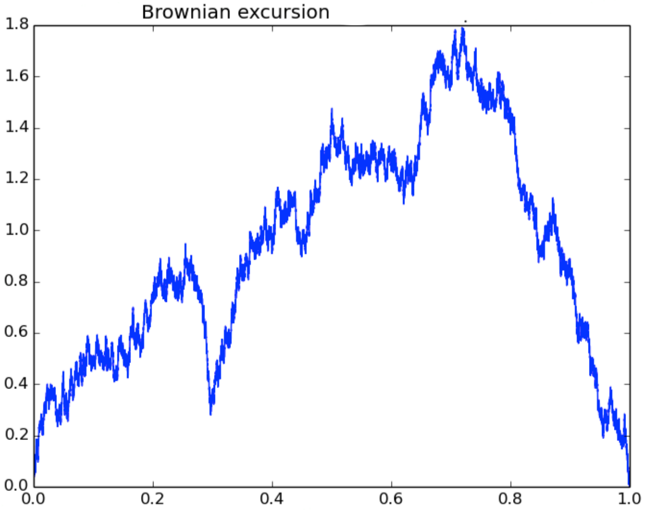
If $e = e(\omega)\colon [0,1] \to [0,\infty)$ is a sample of Brownian excursion, define

$$d(s,t) = e(s) + e(t) - 2 \min_{u \in [s,t]} e(u).$$

Then $T = [0,1]/\sim$, where

$$\begin{aligned} s \sim t :&\Leftrightarrow d(s,t) = 0 \\ &\Leftrightarrow e(s) = e(t) = \min_{u \in [s,t]} e(u). \end{aligned}$$

Brownian excursion



Topology of the CRT

Problem (Curien 2014)

Are two independent samples of the CRT almost surely homeomorphic?

Theorem (B.-Tran 2018)

The CRT is almost surely homeomorphic to the CSST.

Actually, this theorem was essentially proved earlier by Croyden and Hambly in 2008.

For the previous and other results on the topology of trees see:
M. Bonk and Huy Tran, *The continuum self-similar tree*, Preprint, March 2018, on arXiv.

Topology of the CRT

Problem (Curien 2014)

Are two independent samples of the CRT almost surely homeomorphic?

Theorem (B.-Tran 2018)

The CRT is almost surely homeomorphic to the CSST.

Actually, this theorem was essentially proved earlier by Croyden and Hambly in 2008.

For the previous and other results on the topology of trees see:
M. Bonk and Huy Tran, *The continuum self-similar tree*, Preprint, March 2018, on arXiv.

Topology of the CRT

Problem (Curien 2014)

Are two independent samples of the CRT almost surely homeomorphic?

Theorem (B.-Tran 2018)

The CRT is almost surely homeomorphic to the CSST.

Actually, this theorem was essentially proved earlier by Croyden and Hambly in 2008.

For the previous and other results on the topology of trees see:
M. Bonk and Huy Tran, *The continuum self-similar tree*, Preprint, March 2018, on arXiv.

Topology of the CRT

Problem (Curien 2014)

Are two independent samples of the CRT almost surely homeomorphic?

Theorem (B.-Tran 2018)

The CRT is almost surely homeomorphic to the CSST.

Actually, this theorem was essentially proved earlier by Croyden and Hambly in 2008.

For the previous and other results on the topology of trees see: M. Bonk and Huy Tran, *The continuum self-similar tree*, Preprint, March 2018, on arXiv.

Quasisymmetries

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f: X \to Y$ a homeomorphism.
The map f is *quasisymmetric* (=qs) iff there exists a distortion function $\eta: [0, \infty) \to [0, \infty)$ s.t.

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right),$$

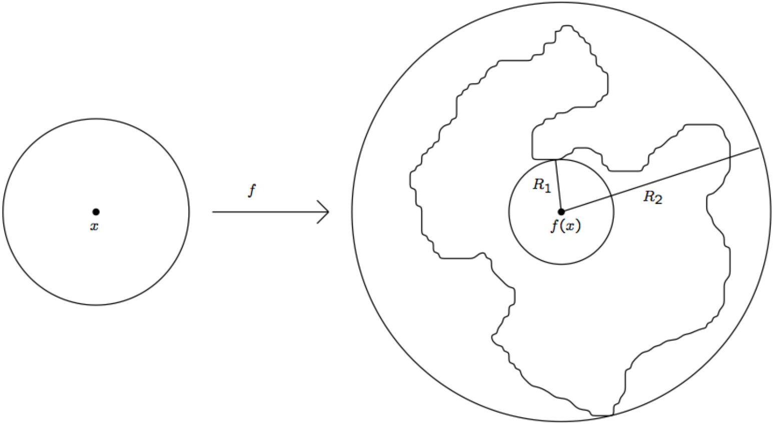
whenever x, y, z are distinct points in X.
If there exists a quasisymmetry $f: X \to Y$, then X and Y are said to be *qs-equivalent*.

Let (X, d_X) and (Y, d_Y) be metric spaces, and $f\colon X \to Y$ a homeomorphism.
The map f is *quasisymmetric* (=qs) iff there exists a distortion function $\eta\colon [0, \infty) \to [0, \infty)$ s.t.

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right),$$

whenever x, y, z are distinct points in X.
If there exists a quasisymmetry $f\colon X \to Y$, then X and Y are said to be *qs-equivalent*.

Geometry of a quasisymmetric map



$$R_2/R_1 \leq \text{Const.}$$

Remarks

- f is quasisymmetric if it maps balls to "roundish" sets of uniformly controlled "eccentricity".
- Quasisymmetry global version of quasiconformality.
- bi-Lipschitz $\Rightarrow$ qs $\Rightarrow$ qc.
- In $\mathbb{R}^n$, $n \geq 2$: qs $\Leftrightarrow$ qc.
  Also true for "Loewner spaces" (Heinonen-Koskela).

Problems about the quasiconformal geometry of trees

- When can one promote a homeomorphism between trees to a quasisymmetry?
- Is there a characterization of the CSST up to qs-equivalence?
- What can one say about the qc-geometry of the CRT or Julia sets of postcritically-finite polynomials?
- Are there some canonical models for certain classes of trees up to qs-equivalence (uniformization problem)?

Qs-characterization of quasi-arcs

Theorem (Tukia-Väisälä 1980)

Let α be a metric arc. Then α is qs-equivalent to $[0,1]$ iff α is doubling and of bounded turning.

A metric space X is *doubling* is there exists $N \in \mathbb{N}$ such that every ball in X can be covered by N (or fewer) balls of half the radius.

A metric space (X, d) is of *bounded turning* if there exists $K > 0$ such that for all points $x, y \in X$ there exists a continuum γ with $x, y \in \gamma$ and

$$\operatorname{diam} \gamma \leq K d(x, y).$$

Qs-characterization of quasi-arcs

Theorem (Tukia-Väisälä 1980)

Let α be a metric arc. Then α is qs-equivalent to $[0,1]$ iff α is doubling and of bounded turning.

A metric space X is *doubling* is there exists $N \in \mathbb{N}$ such that every ball in X can be covered by N (or fewer) balls of half the radius.

A metric space (X, d) is of *bounded turning* if there exists $K > 0$ such that for all points $x, y \in X$ there exists a continuum γ with $x, y \in \gamma$ and

$$\operatorname{diam} \gamma \leq K d(x, y).$$

Qs-characterization of quasi-arcs

Theorem (Tukia-Väisälä 1980)

Let α be a metric arc. Then α is qs-equivalent to $[0,1]$ iff α is doubling and of bounded turning.

A metric space X is *doubling* is there exists $N \in \mathbb{N}$ such that every ball in X can be covered by N (or fewer) balls of half the radius.

A metric space (X, d) is of *bounded turning* if there exists $K > 0$ such that for all points $x, y \in X$ there exists a continuum γ with $x, y \in \gamma$ and

$$\operatorname{diam} \gamma \leq K d(x, y).$$

Qs-uniformization of quasi-trees

Theorem (B.-Meyer)

Let T be a quasi-tree*, i.e., a tree that is doubling and of bounded turning. Then T is qs-equivalent to a geodesic tree.*

A tree (T, d) is *geodesic* if for all $x, y \in T$,

$$\text{length}\,[x, y] = d(x, y).$$

Note: in a quasi-tree we have $\text{diam}\,[x, y] \asymp d(x, y)$.

Proof of Theorem: Decompose the metric space (T, d) into pieces, and carefully redefine metric d by assigning new diameters to these pieces to obtain a geodesic metric ϱ on T. With suitable choices, the identity map $(T, d) \to (T, \varrho)$ is a quasisymmetry. $\square$

Qs-uniformization of quasi-trees

Theorem (B.-Meyer)

Let T be a quasi-tree, *i.e., a tree that is doubling and of bounded turning. Then T is qs-equivalent to a geodesic tree.*

A tree (T, d) is *geodesic* if for all $x, y \in T$,

$$\text{length}\,[x, y] = d(x, y).$$

Note: in a quasi-tree we have $\text{diam}\,[x, y] \asymp d(x, y)$.

Proof of Theorem: Decompose the metric space (T, d) into pieces, and carefully redefine metric d by assigning new diameters to these pieces to obtain a geodesic metric ϱ on T. With suitable choices, the identity map $(T, d) \to (T, \varrho)$ is a quasisymmetry. □

Theorem (B.-Meyer)

Let T be a quasi-tree, *i.e., a tree that is doubling and of bounded turning. Then T is qs-equivalent to a geodesic tree.*

A tree (T, d) is *geodesic* if for all $x, y \in T$,

$$\text{length}\,[x, y] = d(x, y).$$

Note: in a quasi-tree we have $\text{diam}\,[x, y] \asymp d(x, y)$.

Proof of Theorem: Decompose the metric space (T, d) into pieces, and carefully redefine metric d by assigning new diameters to these pieces to obtain a geodesic metric ϱ on T. With suitable choices, the identity map $(T, d) \to (T, \varrho)$ is a quasisymmetry. $\square$

Theorem (B.-Meyer)

Let T be a quasi-tree, *i.e., a tree that is doubling and of bounded turning. Then T is qs-equivalent to a geodesic tree.*

A tree (T, d) is *geodesic* if for all $x, y \in T$,

$$\text{length}\,[x, y] = d(x, y).$$

Note: in a quasi-tree we have $\text{diam}\,[x, y] \asymp d(x, y)$.

Proof of Theorem: Decompose the metric space (T, d) into pieces, and carefully redefine metric d by assigning new diameters to these pieces to obtain a geodesic metric ϱ on T. With suitable choices, the identity map $(T, d) \to (T, \varrho)$ is a quasisymmetry. $\square$

Further results I

Theorem (B.-Tran)

The Julia set $\mathcal{J}(z^2 + i)$ is homeomorphic to the CSST.

Further results I

Theorem (B.-Tran)

The Julia set $\mathcal{J}(z^2 + i)$ is homeomorphic to the CSST.

Proof: Study the the Hubbard tree (=convex hull of critical and postcritical points in Julia set). □

Further results I

Theorem (B.-Tran)

The Julia set $\mathcal{J}(z^2 + i)$ is homeomorphic to the CSST.

Proof: Study the the Hubbard tree (=convex hull of critical and postcritical points in Julia set). □

Theorem (B.-Tran 2015)

The Julia set $\mathcal{J}(z^2 + i)$ is qs-equivalent to the CSST.

Proof: Carefully cut T into pieces with good geometric control and align with pieces of CCST. Geometric control comes from conformal elevator techniques. □

Further results I

Theorem (B.-Tran)

The Julia set $\mathcal{J}(z^2 + i)$ is homeomorphic to the CSST.

Proof: Study the the Hubbard tree (=convex hull of critical and postcritical points in Julia set). □

Theorem (B.-Tran 2015)

The Julia set $\mathcal{J}(z^2 + i)$ is qs-equivalent to the CSST.

Proof: Carefully cut T into pieces with good geometric control and align with pieces of CCST. Geometric control comes from conformal elevator techniques. □

A better proof would follow from a qs-characterization of the CSST (open problem!).

Further results II

Theorem (B.-Tran)

Let P be a postcritically-finite polynomial. If its Julia set $\mathcal{J}(P)$ is homeomorphic to the CSST, then it is qs-equivalent to the CSST.

Further results II

Theorem (B.-Tran)

Let P be a postcritically-finite polynomial. If its Julia set $\mathcal{J}(P)$ is homeomorphic to the CSST, then it is qs-equivalent to the CSST.

Proof: Same ideas. □

Quasiconformal geometry of the CRT

Question

Is the CRT almost surely qs-equivalent to the CSST?

Question

Is the CRT almost surely qs-equivalent to the CSST?

No, because the CSST is a quasi-tree, and in particular doubling, while the CRT is not doubling, and doubling is preserved under quasisymmetries.

Quasiconformal geometry of the CRT

Question

Is the CRT almost surely qs-equivalent to the CSST?

No, because the CSST is a quasi-tree, and in particular doubling, while the CRT is not doubling, and doubling is preserved under quasisymmetries.

Open Problem (very hard!)

Are two independent samples of the CRT almost surely qs-equivalent?

Quasiconformal geometry of the CRT

Question

Is the CRT almost surely qs-equivalent to the CSST?

No, because the CSST is a quasi-tree, and in particular doubling, while the CRT is not doubling, and doubling is preserved under quasisymmetries.

Open Problem (very hard!)

Are two independent samples of the CRT almost surely qs-equivalent?

General theme: Geometric uniqueness of probabilistic models. Known (up to quasi-isometric equivalence) for Bernoulli percolation on $\mathbb{Z}$, Poisson point process on $\mathbb{R}$ (Basu-Sly).